King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

Combination of Orthogonal Collocation Points with Pre-assigned Knots
A Study of Numerical Solutions of Boundary Value Problems

by

M.A. Bokhari and H. Al-Attas

Final Report

Project # FT060002

Submitted to

Deanship of Scientific Research

May 26, 2008

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Acknowledgement

We, the members of the project team, acknowledge with thanks the Research Grant received from the Deanship of Scientific Research for the Project # **FT060002** The library and computing facilities availed during the completion of the project are highly appreciated by us.

Dr. M. A. Bokhari
(Principal Investigator)

Dr. H. Al-Attas
(Co-Investigator)

1. Abstract

1.1 الملخص بالعربي

في هذا العمل تم عرض طريقة التجميع المتعامدة والتى تقرب الحل لمسألة حطية ذات نقطتين ولها قيم حدودية. وخلافا لنقاط جاوسين، لدينا في هذه الطريقة حرية في اختيار نصف عدد نقاط التجميع بإي شكل نريده. ويجري حساب النصف المتبقي على اساس شروط معينة تعامديه. تم بناء هذه الطريقة بتاءاً على فكرة وجود زوج من النقاط المتعامدة (a_1, a_2) في الفترة $[-1,1]$. لقد ناقشنا وجود حلول التجميع بالتفصيل عن طريق دوال قرين. لقد تبين من تحليل الاخطاء أن طريقة النقاط المقترحة تؤدي الى تقارب بمقدار $O(h^{3.5})$ فى حالة حدية. وتشير النتائج العدديه الى أن العديد من الخيارات لa_1 والمقاربة ل -٠،٥٧ توفر أفضل حل تقريبي لمسألة مقارنة بإختيار $a_1 = \frac{-1}{\sqrt{3}} \approx -0.57735027$ ، والتى هى الحالة لنقاط جاوسين. وهذا يدل على أن نقاط جاوسبن كنقاط تجميع ليست بالضروره أفضل خيار في طريقة التجميع المتعامدة.

1.2 Abstract in English

In this work we report an orthogonal collocation method that approximates the solution of a two-point linear boundary value problem. Contrary to the Gaussian nodes, we have a freedom of selecting half of the collocation points arbitrarily in the proposed method. The remaining half are evaluated on the basis of certain orthogonality condition. Construction of these points emerges from a notion of a pair of orthogonal points (a_1, a_2) in $[-1,1]$. We have discussed the existence of collocation solutions in detail via Green's functions. Error analysis show that the proposed points lead the method to $O(h^{3.5})$ convergence in a limiting case. Numerical results indicate that several choices of a_1 in the neighborhood of –0.57 provide a better approximate solution of the BVP versus the choice of $a_1 = \frac{-1}{\sqrt{3}} \approx -0.57735027$, the case of Gaussian knots. This shows that the Gaussian knots as collocation points are not necessarily the best choice in the orthogonal collocation method.

2. Introduction

Several problems in engineering and physical sciences are modeled as differential equations for which the conditions are specified at more than one point. Such models are referred to as boundary value problems (BVP). In particular, if a 2nd order Ordinary Differential Equation (ODE) is subject to certain specific conditions at two points of the independent variables, it is known as a two-point BVP. The ODE involved in a BVP may be linear or nonlinear and homogeneous or non-homogeneous. A second order BVP will be referred to as a two-point BVP. We narrate some problems in the disciplines of engineering and physical sciences which are modeled as two-point BVP.

1. **Convection-diffusion equation** [33]

   $$-D\frac{d^2y}{dx^2}+v\frac{dy}{dx}=S(x); \qquad y(0)=y(1)=0$$

   where the convection coefficient v and the diffusion coefficient D are both positive, and $S(x)$ is a forcing function.

2. **Radial temperature distribution in a cylinder** [73]

   $$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}=0; \qquad T(r_1)=T_1,\ T(r_2)=T_2,$$

   where r_1 and r_2 are the radii corresponding to inner and outer surfaces of the cylinder.

3. **Buckling of a thin vertical column** [73]

   $$EI\frac{d^2y}{dx^2}+Py=0, \qquad y(0)=y(L)=0,$$

   where (i) L is the length of a vertical homogeneous column hinged at both ends and subject to a constant axial load P, (ii) E is young's modulus of elasticity, (iii) I is the moment of inertia of a cross section about a vertical line through its centroid.

4. **Rotating string about the *x*-axis** [73]

$$\frac{d}{dx}\left(T(x)\frac{dy}{dx}\right)+\rho\omega^2 y=0, \qquad y(0)=y(L)=0,$$

where (i) L is the length of string fixed at $x = 0$ and $x = L$ with constant linear density ρ, (ii) $T(x)$ is non-constant magnitude of the tension acting tangential to the string, (iii) ω is angular velocity.

5. **Deflection of a Beam** [42]

$$y''-\frac{T}{EI}y=\frac{wx(x-L)}{2EI}, \qquad y(0)=y(L)=0$$

where E is the modulus of elasticity , I the central moment of inertia and L is length of the beam fixed at $x = 0$ and $x = L$.

Depending on the coefficients or nature of the forcing function of a BVP, sometimes it is difficult to determine its analytic solution. In such situations, a numerical approximation and graphical visualization of a BVP is highly desirable. Our main objective in the present work is to discuss a specific technique "Collocation Method" for the approximation of a two-point linear BVP.

2.1. Two-point linear BVP

The most general form of a two point linear BVP is given by

$$\begin{cases} x''=p(t)x'+q(t)x+r(t), & a<t<b \\ x(a)=\alpha, \quad x(b)=\beta, \end{cases} \tag{2.1}$$

where $p(t)$, $q(t)$ and $r(t)$ are real-valued functions defined on $[a,b]$. A sufficient condition for the existence and uniqueness of the solution of problem (2.1) is given in the following theorem:

Theorem 2.A [36, p.11]: $p(t)$, $q(t)$ and $r(t)$ are real-valued functions defined on $[a,b]$ with $q(t) > 0$ for $a \le t \le b$. Then the BVP (2.1) has a unique solution for each α and β.

Remark 2.1. It may be noted that a BVP may admit a solution even if the sufficient condition(s) of Theorem 2.A are not satisfied as we observe in

Example 2.1: The BVP

$$x'' = \frac{1}{s}x' - \frac{1}{s^2}x + \ln s, \qquad x(1) = \alpha,\ x(2) = \beta$$

has the unique solution $x(s) = c_1 s + (c_2 s + 1)\ln s + 2$ for any real values of α and β where $c_1 = \alpha - 2$ and $c_2 = (\beta - \ln 2 - 2\alpha + 2)/2\ln 2$. Here, we note that $q(s) = \frac{-1}{s^2} < 0$ for $1 \le s \le 2$. (The solution is not correct)

Remark 2.2. **(**[36], [47]**):** Conditions of the form given in (2.1) are known as Dirichlet Boundary Conditions. By contrast, a modified version of the problem (2.1) consists of imposing conditions on the derivative, i.e., $x'(a) = \eta$ and $x'(b) = \mu$. These conditions are known as Neumann Boundary Conditions. A BVP, in some situations, can be described by a combination of Dirichlet and Neumann boundary conditions.

2.2. BVP with homogeneous boundary conditions

The problem (2.1) includes nonzero boundary conditions, i.e., we may have $\alpha \ne 0$ and/or $\beta \ne 0$. In general, it is convenient to deal with homogeneous boundary conditions. Nonzero boundary conditions in (2.1) can easily be converted to homogeneous ones by a simple transformation. For this, set [59]

$l(t) = \frac{1}{b-a}\{\alpha(b-t) - \beta(a-t)\}$. l is a linear function with $l(a) = \alpha$ and $l(b) = \beta$. If $x(t)$ is the solution of the BVP (2.1), we set

$w(t) = x(t) - l(t)$.

Then $w(t)$ is the solution of the transformed BVP

$$\begin{cases} w'' = p(t)w' + q(t)w + R(t), & a < t < b \\ w(a) = 0, \quad w(b) = 0, \end{cases} \tag{2.2}$$

where $R(t)$ is related to the coefficients of the ODE (2.1) as follows:

$$R(t) = r(t) - l(t)q(t) - \frac{\beta - \alpha}{b - a}p(t). \tag{2.3}$$

To see this, it is enough to note that $w'(t) = x'(t) - \frac{\beta - \alpha}{b - a}$ and $w''(t) = x''(t)$. Thus, $x(t)$ is the solution of (2.1) if and only if $w(t)$ is the solution of (2.2).

From now onward we shall consider BVP of the form (2.2) and discuss different ways to determine its approximate solution.

2.3. Numerical techniques

Some of the numerical techniques which are most common to determine approximate solution of the BVP (2.2) are briefly explained below.

1. ***Finite differences method*** (*Direct method*) [37, p.589]. This method is based on the replacement of 1st and 2nd derivatives by the finite differences with respect to the mesh created on the underlying interval $[a,b]$. The resulting approximation of the derivatives leads to a system of linear algebraic equations (or nonlinear in case BVP is nonlinear). The solution of the system provides a discrete approximation of the solution of the BVP. We shall consider this method for the sake of comparison with our proposed methods.

2. ***Function space approximation methods*** (*Direct method*) [36, p.173]. These techniques also provide discrete approximation of the solution of a BVP and involve tremendous variety in the choice of approximating functions. Among these, three procedures are quite prominent. They are
    - Galerkin's method,
    - Ritz method,
    - Method of collocation.

3. ***Shooting method*** (*Indirect Method*) [37, p.581]. This is an indirect approach to solve a BVP (2.2). Here, the BVP is converted to an appropriate initial value problem (IVP). The resulting IVP is solved with a guess as to the appropriate initial value $w'(a)$. The associated ODE is then integrated to obtain an approximate solution with the hope that $w(a) = 0$. If $w(a) \neq 0$, then the guessed

value of $w'(a)$ is changed and the method is repeated till the iterative solutions converge to the solution of the given BVP.

We shall focus our attention on collocation methods which is a theme of our project. In order to explain these methods in detail we shall require operator form of linear BVP.

2.4. Operator form of BVP

The problem (2.2) may be formulated somewhat differently by use of a second degree linear differential operator. For this, we consider the *solution space*

$$X = \{\varsigma \in C^2[a,b] : \varsigma(a) = \varsigma(b) = 0\} \tag{2.4}$$

for the problem (2.2) and define $L : X \to C[a,b]$ as $L(\varsigma) = \dfrac{d^2\varsigma}{dt^2} - p(t)\dfrac{d\varsigma}{dt} - q(t)\varsigma,\ \varsigma \in X$.

The BVP (2.2) is thus equivalent to solving the equation

$$L(\varsigma) = R, \qquad \varsigma \in X \ . \tag{2.5}$$

Equation (2.5) will be referred to as the operator form of the BVP (2.2). It may be noted that *X* is a real vector space and *L* is a linear operator on *X*. We are interested to find an approximate solution of (2.5) by the collocation methods. Their brief description is given below (*Also see Chapter 3*).

2.5. Collocation Methods (CM)

An approximate solution of (2.5) by collocation methods is a function *u* of the form

$u(t) = \sum_{i=1}^{N} c_i u_i(t)$ where $\{u_i : i = 1,2,...,N\}$ is a basis of an approximating *N*-dimensional subspace X_N of *X* (cf (2.4)). In order to determine the *N* parameters c_i's, we select *N* *collocation points* $t_i \in (a,b),\ i = 1,2,...,N$, and consider the constraints:

$$(Lu)(t_j) = R(t_j),\ j = 1,2,...,N \ . \tag{2.6}$$

Finding solution of (2.6) is equivalent to solving the following system of linear equations

$$\sum_{i=1}^{N} c_i L(u_i)(t_j) = R(t_j),\ j = 1,2,...,N$$

(2.7)

for the unknowns $c_i : i = 1,2,...,N$.

2.6. Orthogonal Collocation Methods (OCM)

There is a general drawback in the CM if the knots t_i's in (2.7) are equally spaced. It is known that accuracy due to CM is of order h^2 whereas the Finite Element Method (FEM) when applied to the problem (2.2) provides an order h^4 accuracy [30]. This disparity in the rate of convergence is related to the choice of the collocation knots appearing in the system of linear equations (2.7). On the other hand, the collocation at the Gaussian knots (i.e., the zeros of shifted Legendre polynomials) speeds up the rate of convergence and makes it competitive with Galerkin Methods [49]. Collocation in this context is referred to as the Orthogonal Collocation Method (OCM). Here, we consider a partition of the interval [0,1]:

$$0 = x_0 < x_1 < x_2 < ...... < x_k = 1, \tag{2.8}$$

and fix two Gaussian knots $t_{i1}, t_{i2} \in (x_{i-1}, x_i), i = 1,2,....,k$ [1]. Then collocation at the $2k$ knots using a basis that involves piecewise cubic Hermite polynomials provides an order h^4 convergence for the problem (2.2) [29].

2.7. Theme of the project

The main objective of the project is to discuss OCM by introducing some freedom in the selection of collocation knots while keeping them zeros of polynomials orthogonal in some sense over the underlying interval. Recall that the pairs of collocation points considered in the OCM are the pairs of the zeros of shifted Legendre polynomial. These points are fixed by default. In our work, we shall fix a point t_{i1} in $[-1,1]$ and determine another point t_{i2} such that both points turn up as the distinct zeros of certain 2nd degree polynomial which will be orthogonal in some sense over $[-1,1]$. Such polynomials will

[1] The two knots are the zeros of 2nd degree shifted Legendre Polynomial in the interval $[x_{i-1}, x_i]$.

be referred to as “Orthogonal 0-interpolants” (or “Interpolating orthogonal polynomials” as indicated in [6]). In this report, we shall focus on

1. the OCM relevant to the zeros of the “Orthogonal 0-interpolants” (we shall name the resulting method “*Interpolating Orthogonal Collocation Methods* (*IOCM*)”.
2. computational procedure for the IOCM.
3. exploring the condition that may guarantee the availability of pairs of distinct zeros (t_{i1}, t_{i2}) in each subinterval $[x_{i-1}, x_i]$ (cf (2.8)).
4. MATLAB based implementation of the *IOCM* to selective BVP’s with known solutions and comparison of the resulting solutions with those obtained by
   i. OCM (based on Gaussian knots).
   ii. FDM (Finite Difference Method)
   iii. SM (Shooting Method)
5. Relationship between (t_{i1}, t_{i2}) and the pair of Gaussian knots in $[x_{i-1}, x_i]$.
6. Detailed explanation on the existence of collocation solution
7. Error analysis based on *IOCM*.

2.8. Literature survey

There is a wide range of literature on the theoretical and computational aspects as well as on applications of OCM to engineering problems. However, we could not trace any material which refers to the choice of “Collocation Points” as proposed by us. A thorough review of literature paved our way towards comparative study between our proposed method IOCM with OCM (based on Gaussian knots), FEM and shooting method. Below, we present a few articles that deal with basics as well as some notable developments and applications related to OCM.

Quarteroni and Valli [50] and afterward Karniadakis and Sherwin [34] described in their books that the collocation methods represent an economical alternative since they only requires the evaluation of these terms at grid points. This method has been most successfully applied in the past to orthogonal polynomial approximations since the early work of Rubin and Graves [50].

Carey and Oden [10] regarded the collocation method as a member of the finite element methods.

Russell and Christiansen [56] and Ascher et al [1] established that the OCM's have an advantage of having a known optimal algorithm for placing the mesh points. These advantages make the methods very suitable for solving difficult linear and nonlinear problems.

De Boor and Swartz [15] pointed out that orthogonal collocation yields optimal order accuracy for the error. In [13], De Boor showed that using piecewise polynomials is more effective in representing the solutions to the differential equations than pure polynomials, i.e., it is more effective to fit polynomials to smaller segments of the underlying intervals/regions of the solution than to fit polynomials to the entire domain of the solution. In the same article, he showed that how to construct and apply B-spline curve to form the solution of an ODE.

Robin and Graves [51] were the early developers of a cubic spline collocation schemes for solving fluid flow equations. Fairword and Meade [17] gave an extensive review of spline collocation methods and their applications to various problems. They classified the spline collocation into four categories: nodal, orthogonal, extrapolated/modified and collocation/galerkin.

Various aspects of spline collocation were considered in different directions by Botella [8], Farin [18], Johnsaon [33] and Sun [63]. In particular, Farin [18] compared the applications of nodal and orthogonal collocation as well as the collocation at Greville abscissa ([28], [29]) by considering cubic and quartic B-spline curves of various levels of continuity.

Seoane [61] obtained approximation results for isolated centers and perturbed bifurcation points of nonlinear differential equations depending on two parameters when the problem was discretized by OCM. An alternating direction implicit method is analyzed by Bialecki [2] for the solution of linear systems arising in higher-order, tensor-product orthogonal spline collocation applied to some separable, second order, linear, elliptic PDE's in rectangles.

Bialecki and Fairwehther [5] review the use of orthogonal collocation for various bivariate linear PDE. Prenter and Russell has described $O(h^4)$ OCM for approximating the solution of elliptic PDE's on the unit square [48]. Bialecki [3] with Fairweather and Bennet proposed fast direct menthod for the solution of linear systems arising when orthogonal spline collocation with piecewise Hemite bicubics is employed for the approximate solution of Poisson's equation in a rectangle.

On the application side, we note that Oh and Luus [46] presented a computational procedure that involves OCM for the solutions of two point BVP that emerged from the application of Pontrjagin's maximum principle to systems with unsaturated inputs. Burka [9] applied OCM to develop a fast and accurate method for the integration of large sparse systems of stiff initial value ordinary differential equations. Johnson [33] has applied a B-spline collocation method to the solution of the 1-D steady, linear 2nd order, convection-diffusion equation using quartic B-splines with collocation at the Greville abscissa. Using OCM, Wysocki provided the numerical treatment of control systems described by first order hyperbolic equations [71]. Mohsen and Pinder devised a finite element scheme using OCM and applied it to linearized Burgers' problem [45]. Lee [41] used OCM to solve the radial temperature and velocity distributions in the one-dimensional arc. Applying OCM , Tarn and Lin [64] extended the Finite Strip Method to the solution of BVP's involving nonrectangular domains. A technique to construct a low-order finite difference preconditioner for solving orthogonal collocation equations for BVP's is presented by Sun et al [62]. Rui et al [54] used OCM in solving nonlinear dynamic games. Ganesh et al [26] applied OCM to second order nonlinear Fredholm integro differential equations subject to Neumann boundary conditions. Edoh et al [16] presented an $O(h^4)$ OCM for solving nonlinear first order PDE's with periodic boundary conditions. Cizniar et al [11] developed OCM on finite elements and implemented it within MATLAB environment for dynamic optimization. In [52], Razzaghi et al provided a numerical method for solving linear quadratic optimal control problems in which inequality constraints are first converted to a system of algebraic equalities; these equalities are then collocated at Legendre-Gauss-Lobatto nodes. Bialecki et al [4] discussed orthogonal spline collocation solution of Dirichlet BVP for Poison's equation on an L-shaped region.

2.9. Organization of report

The distribution of material in this report is as follows: Chapters 3 to 11 provide the main content of the report. While surveying the literature, we were unable to trace an article or a monograph that fully justifies the topic. Therefore, we incorporated some essentials on the theory of two-point BVP in the report to benefit a beginner on this topic. Chapter 3 explains some notions related to collocation points and basic polynomials required in the process of CM. Also, approximate solutions with different choices of collocation points and approximating polynomials for a simple BVP are compared with the respective solution due to FDM.

Existence of collocation solution is discussed in Chapter 4. Here, we have included some material on explicit solution of BVP (2.2) in terms of the Green's function as well as a correspondence between the solutions of a Fredholm integral equation and a BVP.

The notion of the orthogonal 0-interpolants is introduced in Chapter 5 where as Chapter 6 deals with the construction of piecewise cubic Hermite polynomials.

In Chapter 7 we have explained OCM. A detailed discussion on "Error Analysis" is given in Chapter 8 which also includes our main result.

Computational procedure is explained in Chapter 9 and its implementation to some selective examples in Chapter 10. The simulation results related to the examples are obtained in the MATLAB environment.

In Chapter 11, we indicated some further directions of the work. Because of certain limitations we restricted our work to BVP based on ODE and could not tackle the PDE based problems.

3. Method of Collocation

The method of collocation is a strategy to determine approximate solution of the operator equation

$$L\varsigma = f \tag{3.1}$$

described (2.5). Here, f is known and ς is to be sought. A general description of the method is given below.

3.1. Description of the method of collocation

Recall that the operator $L : X \to C[a,b]$ as considered in Section 2.4 is linear and X, the solution space of (2.2), is a real vector space. In order to find an approximate solution of $L\varsigma = f$ from an N- dimensional subspace $X_N \subset X$, we proceed as follows:

i. Select a suitable basis, $\{\phi_1, \phi_2, \ldots, \phi_N\}$ of X_N.

ii. Consider ς_N, a linear combination of basis functions $\phi_1, \phi_2, \ldots, \phi_N$, i.e.,

$$\varsigma_N = c_1\phi_1 + c_2\phi_2 + \cdots + c_N\phi_N \tag{3.2}$$

where c_i's are unknowns, and replace ς by ς_N in (3.1). Because of linearity of L, we arrive at a system of linear equations in N unknowns $c_i, 1 \le i \le N$:

$$\sum_{i=1}^{N} c_i L\phi_i = f \tag{3.3}$$

iii. In order to determine the unknowns, we select N distinct points $t_1, t_2, \ldots, t_N \in [a,b]$ and consider the "Collocation Equation"

$$\sum_{i=1}^{N} c_i L\phi_i(t_j) = f(t_j), \qquad j = 1, 2, \ldots, N. \tag{3.4}$$

A unique solution, say "$c_1^*, c_2^*, \ldots, c_N^*$", of the system (3.4) exists if its coefficient matrix $\left[L\phi_i(t_j)\right]_{i,j=1}^{N}$ is invertible.

iv. Find the unique[2] solution, say, "$c_1^*, c_2^*, \ldots, c_N^*$" of the linear system (3.4) and set

$$\varsigma_N = c_1^* \phi_1 + c_2^* \phi_2 + \cdots + c_N^* \phi_N .$$

Then ς_N is the approximate solution for the operator equation (3.1) and also for the BVP (2.2).

Note: The points $t_1, t_2, \ldots, t_N$ in (3.4) are referred to as Collocation Points or Collocation knots.

3.2. Impact of basic functions and collocation points

Suppose that exact solution of (3.1) given by ς is approximated by $\varsigma_N^* = \sum_{i=1}^{N} c_i^* \phi_i$ with pre-assigned collocation points t_i , $1 \le i \le N$. In order to determine the level of accuracy of approximate solution ς_N^* , we consider standard types of errors:

Definition 3.1. Pointwise error "Err" with respect to collocation points t_i , $1 \le i \le N$, is defined as

$$\text{Err}\,(t_i) = \varsigma(t_i) - \varsigma_N^*(t_i), \qquad i = 1, 2, \ldots, N . \tag{3.5}$$

Definition 3.2. Maximum error "M-Err" with respect to collocation points t_i , $1 \le i \le N$, is given by

$$\text{M-Err} = \max_{1 \le i \le N} \left| \varsigma(t_i) - \varsigma_N^*(t_i) \right| . \tag{3.6}$$

Definition 3.3. Root Mean squared error "RMS-Err" with respect to collocation points t_i , $1 \le i \le N$, is defined as

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{N} \left| \varsigma(t_i) - \varsigma_N^*(t_i) \right|^2}{N}} . \tag{3.7}$$

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[2] Note that in the coefficient matrix $\left[L\phi_i(t_j) \right]_{i,j=1}^{N}$, $\phi_1, \phi_2, \ldots, \phi_N$ are linearly independent and the points $t_1, t_2, \ldots, t_N$ are distinct. Therefore, the coefficient matrix is non-singular.

The choice of basic functions ϕ_i and the collocation points t_i , $1 \le i \le N$, affects the approximation errors (3.5)-(3.7). This phenomenon is explained with the help of 2nd order BVP (see Example 3.1 below) by considering different sets of collocation points and two sets of slightly different basic functions. We apply the procedure of Section 3.1 to this example for $N = 3$ and compare the error of approximation based on different collocation points and basic functions. The errors determined by CM are also compared with those arising from the approximate solution due to the finite difference method.

Example 3.1

BVP: $$y'' - y = \cos t, \qquad y(0) = y(1) = 0 \tag{3.8}$$

Exact Solution: $$y(t) = \frac{e^{-1} - \cos(1)}{2(e^{-1} - e)} e^{t} + \frac{\cos(1) - e}{2(e^{-1} - e)} e^{-t} - \frac{1}{2}\cos t\,. \tag{3.9}$$

Sets of collocation points.

Set P-1: Uniformly distributed points:

$$t_1 = 0, \qquad t_2 = 0.5, \qquad t_3 = 1. \tag{3.10}$$

Set P-2: (***Shifted Legendre Triplets***) Shifted zeros of 3rd degree Legendre polynomial

$$t_1 = \frac{1}{2}, \qquad t_2 = \frac{\left(1 - \sqrt{3/5}\right)}{2},\ t_3 = \frac{\left(1 + \sqrt{3/5}\right)}{2}. \tag{3.11}$$

These zeros are obtained by means of the linear function $\lambda : [-1,1] \to [0,1]$ defined as $\lambda(t) = \frac{1}{2}(t+1)$ for $t = 0, \pm\sqrt{\frac{3}{5}}$

Set P-3: (***Orthogonal Triplets***) Zeros of 3rd degree orthogonal 0-interpolants (*OZI*) [11] on [0,1] for which first zero t_1 is selected anywhere in [0,1] except the midpoint of the interval. Other zeros are determined by using orthogonality condition as explained in Section 4.2, e.g.,

$$\left.\begin{array}{llll} \text{(i) } t_1 = 0 & \Rightarrow & t_2 = 0.45585, & t_3 = 0.87749 \\ \text{(ii) } t_1 = 0.75 & \Rightarrow & t_2 = 0.12662, & t_3 = 0.66825 \end{array}\right\}, \tag{3.12}$$

Sets of approximating functions.

Set F-1:

$$\phi_1(t) = t(1-t), \qquad \phi_2(t) = t^2(1-t), \qquad \phi_3(t) = t^2(1-t)^2 . \tag{3.13}$$

Here, $L\phi_1(t) = -2 - t(1-t)$, $L\phi_2(t) = 2 - 6t - t^2(1-t)$, $L\phi_3(t) = 2 - 12t + 12t^2 - t^2(1-t)^2$.

Set F-2:

$$\phi_1(t) = t(1-t), \qquad \phi_2(t) = t^2(1-t), \qquad \phi_3(t) = t^3(1-t) . \tag{3.14}$$

Here, $L\phi_1(t) = -2 - t(1-t)$, $L\phi_2(t) = 2 - 6t - t^2(1-t)$, $L\phi_3(t) = 6t - 12t^2 - t^3(1-t)$.

Approximate Solutions

I. Structure. The general form of the approximate solution based on collocation points t_1, t_2, t_3 and basic functions ϕ_1, ϕ_2, ϕ_3 is given by

$$\varsigma_3(t) = c_1^* L\phi_1(t) + c_2^* L\phi_2(t) + c_3^* L\phi_3(t) \tag{3.15}$$

where c_1^*, c_2^*, c_3^* satisfy the linear system:

$$\begin{bmatrix} L\phi_1(t_1) & L\phi_2(t_1) & L\phi_3(t_1) \\ L\phi_1(t_2) & L\phi_2(t_2) & L\phi_3(t_2) \\ L\phi_1(t_3) & L\phi_2(t_3) & L\phi_3(t_3) \end{bmatrix} \begin{bmatrix} c_1^* \\ c_2^* \\ c_3^* \end{bmatrix} = \begin{bmatrix} \cos t_1 \\ \cos t_2 \\ \cos t_3 \end{bmatrix}.$$

II. Legend. We shall denote the approximate solution (3.16) by $\varsigma_{3,(i,j)}$ if the collocation points are based on Set P-1, P-2 and P-3 (cf (3.10)-(3.12)). The approximating functions are related to the Set F-1 and F-2 (cf (3.13)-(3.14)). For example, the use of collocation points from Set P-1 and the approximating functions from Set F-1 results to the approximate solution

$$\varsigma_{3,(1,1)}(t) = -0.42675\, t(1-t) + 0.07662 t^2(1-t) - 0.00337 t^2(1-t)^2 . \tag{3.16}$$

Similarly, for Sets P-1 and F-2, we have

$$\varsigma_{3,(1,2)}(t) = -0.43138 t(1-t) + 0.068619 t^2(1-t) - 0.0026657 t^3(1-t) . \tag{3.17}$$

III. Computational results.

1. Using (3.9), the errors of approximation due to (3.16) and (3.17) are calculated. Also, we determined the approximation error due to the finite difference method (FDM) with uniform mesh size $h = 0.1$. The results are summarized below in Table 3.1.

Table 3.1: Point-wise & RMS-Errors based on F-1 & F-2 functions using P-1 points

t	**Point-wise Error** ***FDM*** Mesh size: $h = 0.1$	**Point-wise Error** ***CM***	
		(P-1,F-1)	**(P-1,F-2)**
0.0	0	0	0
0.1	2.4691e-006	1.0313e-005	4.7384e-004
0.2	4.6510e-006	1.6981e-005	9.4403e-004
0.3	6.3329e-006	2.0156e-005	1.3975e-003
0.4	7.3728e-006	2.1639e-005	1.8081e-003
0.5	7.6951e-006	2.2974e-005	2.1354e-003
0.6	7.2857e-006	2.4403e-005	2.3227e-003
0.7	6.1883e-006	2.4659e-005	2.2976e-003
0.8	4.5006e-006	2.1528e-005	1.9722e-003
0.9	2.3718e-006	1.3148e-005	1.2444e-003
1.0	0	0	0
***RMS* Err**	**1.7282e-005**	**6.0307e-005**	**5.1971e-003**
***M* Err**	**7.6951e-006**	**2.4659e-005**	**2.3227e-003**

2. The set of shifted Legendre zeros as collocation points (cf P-2) leads to the following approximate solutions.

$$\varsigma_{3,(2,1)}(t) = -0.42664t(1-t) + 0.07664t^2(1-t) - 0.00365t^2(1-t)^2$$

$$\varsigma_{3,(2,2)}(t) = -0.43179t(1-t) + 0.06805t^2(1-t) - 0.00293t^3(1-t).$$

Respective approximation errors are given in the following Table 3.2.

3. In the following table, we provide the errors of approximation by considering the zeros of orthogonal 0-interpolantsIts (see Set P-3). The location of first zero t_1 in P-3 has an impact on the approximation error. Here, we note that the choice of $t_1 = 0.3$ along with the polynomials form F-3 gives a better result. For further comparison see Tables 3.3 A and 3.3 B.

Table 3.2: (Comparison of FDM with CM based on Shifted Legendre Zeros)

Point-wise & RMS-Errors based on F-1 & F-2 functions using P-2 points

t	**Point-wise Error** ***FDM*** Mesh size: $h = 0.1$	**Point-wise Error** ***CM*** **(P-2,F-1)**	**(P-2,F-2)**
0.0	0	0	0
0.1	2.4691e-006	1.8435e-006	5.1592e-004
0.2	4.6510e-006	4.5622e-006	1.0292e-003
0.3	6.3329e-006	6.1619e-006	1.5238e-003
0.4	7.3728e-006	6.9596e-006	1.9706e-003
0.5	7.6951e-006	7.6737e-006	2.3246e-003
0.6	7.2857e-006	8.3848e-006	2.5249e-003
0.7	6.1883e-006	8.3225e-006	2.4936e-003
0.8	4.5006e-006	6.4312e-006	2.1368e-003
0.9	2.3718e-006	2.6702e-006	1.3461e-003
1.0	0	0	0
RMS **Err**	**1.7282e-005**	**1.8909e-005**	**5.6485e-003**
M **Err**	**7.6951e-006**	**2.0139e-005**	**2.5687e-003**

Table 3.3 A: (Comparative study of CM based on different Orthogonal Triplets)

Point-wise & RMS-Errors for F-1 & F-2 functions using P-3 points with different t_1

	Collocation Points: $t_1 = 0.0$, $t_2 = 0.45585$, $t_3 = 0.87749$		**Collocation Points:** $t_1 = 0.1$, $t_2 = 0.49903$, $t_3 = 0.8871$		**Collocation Points:** $t_1 = 0.3$, $t_2 = 0.23246$, $t_3 = 0.87315$	
t	**F-1**	**F-2**	**F-1**	**F-2**	**F-1**	**F-2**
0.0	0	0	0	0	0	0
0.1	9.6726e-006	5.1918e-004	2.5737e-006	5.1600e-004	-2.4419e-006	5.4425e-004
0.2	1.5116e-005	1.0331e-003	5.5882e-006	1.0292e-003	-1.5913e-006	1.0868e-003
0.3	1.6193e-005	1.5266e-003	7.1989e-006	1.5238e-003	-3.2762e-007	1.6095e-003
0.4	1.4737e-005	1.9708e-003	7.8454e-006	1.9705e-003	9.7252e-007	2.0807e-003
0.5	1.2643e-005	2.3218e-003	8.3413e-006	2.3244e-003	2.5263e-006	2.4528e-003
0.6	1.0829e-005	2.5192e-003	8.8354e-006	2.5246e-003	4.1050e-006	2.6615e-003
0.7	9.0194e-006	2.4859e-003	8.5980e-006	2.4932e-003	4.8206e-006	2.6256e-003
0.8	6.3172e-006	2.1289e-003	6.5869e-006	2.1365e-003	3.6923e-006	2.2474e-003
0.9	2.4982e-006	1.3405e-003	2.7478e-006	1.3459e-003	9.4575e-007	1.4142e-003
1.0	0	0	0	0	0	0
RMS **Err**	**3.4729e-005**	**5.6389e-003**	**2.0585e-005**	**5.6479e-003**	**8.3992e-006**	**5.9533e-003**
M **Err**	**1.6193e-005**	**2.5192e-003**	**8.8354e-006**	**2.5246e-003**	**4.8206e-006**	**2.6615e-003**

Table 3.3 B: (Comparative study of CM based on different Orthogonal Triplets)

Point-wise & RMS-Errors for F-1 & F-2 functions using P-3 points with different t_1

	Collocation Points: $t_1 = 0.75$ $t_2 = 0.12662$, $t_3 = 0.66825$		**Collocation Points:** $t_1 = 0.90$ $t_2 = 0.11290$, $t_3 = 0.50097$		**Collocation Points:** $t_1 = 1.0$ $t_2 = 0.12251$, $t_3 = 0.54415$	
t	**F-1**	**F-2**	**F-1**	**F-2**	**F-1**	**F-2**
0.0	0	0	0	0	0	0
0.1	-1.1195e-006	5.1790e-004	1.9393e-006	5.1155e-004	1.2424e-006	4.6276e-004
0.2	-8.9574e-007	1.0329e-003	4.7477e-006	1.0204e-003	3.3681e-006	9.2267e-004
0.3	-1.5929e-006	1.5293e-003	6.4934e-006	1.5110e-003	5.2436e-006	1.3671e-003
0.4	-2.9689e-006	1.9777e-003	7.5166e-006	1.9542e-003	7.5818e-006	1.7706e-003
0.5	-4.1851e-006	2.3332e-003	8.5203e-006	2.3057e-003	1.1032e-005	2.0932e-003
0.6	-4.8456e-006	2.5345e-003	9.5310e-006	2.5048e-003	1.5140e-005	2.2791e-003
0.7	-5.2106e-006	2.5034e-003	9.6851e-006	2.4741e-003	1.8137e-005	2.2565e-003
0.8	-5.6305e-006	2.1455e-003	7.7952e-006	2.1206e-003	1.7504e-005	1.9386e-003
0.9	-5.2455e-006	1.3518e-003	3.6499e-006	1.3361e-003	1.1269e-005	1.2241e-003
1.0	0	0	0	0	0	0
***RMS* Err**	**1.1865e-005**	**5.6702e-003**	**2.1361e-005**	**5.6034e-003**	**3.4801e-005**	**5.0984e-003**
***M* Err**	**5.6305e-006**	**2.5345e-003**	**9.6851e-006**	**2.5048e-003**	**7.5818e-006**	**2.2791e-003**

3.3. Some remarks on the choice of collocation points

We observe from the above tables that

i. the performance of CM depends on the choice of collocation points and the basis functions. It gives a smaller error when functions of type F-1 and the collocation points of the form P-3 are used with t_1 = 0.3 or 0.75. This choice gives a better result when compared with FDM or the CM with shifted Legendre zeros.

ii. among the collocation points P-1 (uniformly distributed points) and P-2 (the zeros of shifted 3^{rd} degree Legendre polynomial), P-2 performs better to P-1 for both types of basic functions given in F-1 and F-2.

iii. The choices t_1 = 0.3, 0.75 in P-3 along with F-1 performs better to other choices made for t_1 in P-3.

3.4. Motivation to study CM based on orthogonal pairs or triplets

We note from the above remarks that accuracy of collocation method depends on the choice of collocation points and the underlying basis of a finite dimensional approximating space. Collocation at the zeros of shifted Legendre polynomials does not have any advantage when used with either of F-1 or F-2 type functions.

The most important observation is related to the zeros of 3^{rd} degree orthogonal 0-interpolants (the case of P-3) along with F-1. This combination sometimes outperforms the finite difference method. This observation motivates us to consider the collocation methods based on the orthogonal pairs or triplets.

4. Existence of Collocation Solution

We shall discuss the existence of collocation solution of the two-point BVP:

$$\begin{cases} u'' = P(t)u' + Q(t)u + R(t), & a < t < b, \\ u(a) = 0, & u(b) = 0, \end{cases} \tag{4.1}$$

in this chapter, which will be based on the Green's function approach.

Based on the hypothesis of Theorem 3.A, we assume that Problem (4.1) has a unique solution $x \in X = \{\varsigma \in C^2[a,b] : \varsigma(a) = \varsigma(b) = 0\}$. Recall that solving the BVP (4.1) is equivalent to solving the operator equation $L(\varsigma) = R, \varsigma \in X$ where the differential operator $L : X \to X$ is defined as (cf 2.5)

$$L(\varsigma) = \frac{d^2\varsigma}{dx^2} - P(x)\frac{d\varsigma}{dx} - Q(x)\varsigma, \qquad \varsigma \in X . \tag{4.2}$$

The key to our method is showing that finding the solution of operator equation $L(\varsigma) = R$, $\varsigma \in X$ (cf (4.1)-(4.2)) is equivalent to finding the solution of a specific integral equation.

4.1 Solution of BVP in term of Green's function

Let $w_1(t)$ and $w_2(t)$ be two linearly independent solutions of the homogeneous ODE associated with the BVP (4.1). Then we have

Theorem 4.A. [46, p 34-36]. The solution $w(t)$ of the BVP (4.1) is given by

$$w(t) = \int_a^b G(t,s)R(s)ds \tag{4.3}$$

where

$$G(t,s) = \begin{cases} \dfrac{w_1(s)w_2(t)}{W(s)}, & a < s \le t \\ \dfrac{w_1(t)w_2(s)}{W(s)}, & t < s \le b \end{cases} \tag{4.4}$$

with

$$W(s) = \begin{vmatrix} w_1(s) & w_2(s) \\ w_1'(s) & w_2'(s) \end{vmatrix}. \tag{4.5}$$

Definition 4.1. The function $G(t,s)$ given by (4.4) is called the Green's function associated with the differential operator L (cf (4.2)) with the Dirichlet boundary conditions $\varsigma(a) = \varsigma(b) = 0$.

Remark 4.1. It may be noted that Green's function $G(t,s)$ as given by (4.4) is independent of the forcing function $R(t)$. In other words, the Green's function only depends on $P(t)$ and $Q(t)$ which are the coefficients of the homogeneous equation associated with the BVP (4.1).

Remark 4.2. In case of availability of the Green's function for the two-point BVP of type (4.1), the solution of the BVP reduces to evaluation of the quadrature (4.3). Thus, numerical schemes are sought for approximation of the integral (4.3), in particular, when $R(t)$ is a complicated function or it is available through a discrete approximation. This type of numerical scheme is useful if we are required to solve many problems with the same differential operator and the boundary conditions but with different forcing function $R(t)$. Such a situation arises in different applications, for example, solving the Navier Stokes equations in the infinite channel [28].

There is a general drawback in numerical schemes when applied to (4.3). An application of Gaussian quadrature is disappointing as it leads to slow convergence because of the discontinuities of the Green's function, e.g., see Remark 4.6 (iii). McBain and Armfield [43] noted that rapid convergence may be recovered by the use of product integration [12, p 87]. It requires prior accurate evaluation of the Green's integral for a set of basis functions by a matrix-vector multiplication. This method is quite competitive with OCM ([63], [65]), in particular, when the forcing function $R(t)$ is oscillatory.

Definition 4.2. The operator $\boldsymbol{G}$: $C[a,b] \to C[a,b]$ defined as

$$\boldsymbol{G}(R) = \int_a^b G(.,s)R(s)ds, \qquad \forall R \in C[a,b]. \tag{4.6}$$

will be called integral operator associated with the Green's function $G(t,s)$.

Remark 4.3. The operator $\boldsymbol{G}$ as defined in (4.6) is compact, i.e, any bounded sequence $\{v_n\}$ in $C[a,b]$, has a subsequence $\{v_{n_k}\}$ for which $\{\boldsymbol{G}(v_{n_k})\}$ is convergent in $C[a,b]$.

4.2 BVP vs integral equation

We shall show that solving BVP (4.1) is equivalent to solving a "specific" Fredholm equation of 2nd kind. For this, we proceed with the following lemma:

Lemma 4.A.(*Green's function related to* $u''(t)=v(t)$) Consider the 2nd order differential equation

$$u''(t)=v(t) \text{ subject to } u(a)=u(b)=0. \tag{4.7}$$

Let $X=\{\varsigma\in C^2[a,b]:\varsigma(a)=\varsigma(b)=0\}$ and let $L_0:X\to C[a,b]$ be defined as $L_0(u)=u''$, $u\in X$. Then $g_0(s,t)$, the Green's function corresponding to L_0, is given by

$$g_0(t,s)=\begin{cases}(t-s)-\dfrac{(t-a)(b-s)}{(b-a)} & a\le s\le t\\ -\dfrac{(t-a)(b-s)}{(b-a)} & t<s\le b.\end{cases}$$

Proof. The proof of the Lemma is given on the line of some explanations provided in [48, p. 287]. Integrating $u''(t)=v(t)$ over the interval $[a,s]$, $s\in(a,b]$ leads to

$$u'(s)-u'(a)=\int_a^s u''(\tau)d\tau=\int_a^s v(\tau)d\tau.$$

This equation can be written as $u'(s)=u'(a)+\int_a^s v(\tau)d\tau$. Integrating it over $[a,t]$ we get

$$u(t)-u(a)=\int_a^t u'(a)ds+\int_a^t\left[\int_a^s v(\tau)d\tau\right]ds.$$

Since $u(a)=0$, $u(t)=u'(a)(t-a)+\int_a^t\left[\int_a^s v(\tau)d\tau\right]ds$. Note that R, the region of integration in the double integral is right triangular in $s\tau$ - coordinates and can be expressed in two different ways: $R=\{(s,\tau):a\le\tau\le s;a\le s\le t\}=\{(s,\tau):a\le\tau\le t;\tau\le s\le t\}$. Interchanging the order of integration and then exchanging τ with s, we have

$$\int_{s=a}^{t}\left[\int_{\tau=a}^{s}v(\tau)d\tau\right]ds=\int_{a}^{t}\left[\int_{\tau}^{t}v(\tau)ds\right]d\tau=\int_{a}^{t}(t-\tau)v(\tau)d\tau=\int_{a}^{t}(t-s)v(s)ds\,.$$

Thus,

$$u(t)=u'(a)(t-a)+\int_{a}^{t}(t-s)v(s)ds \tag{4.8}$$

Taking t = b in (4.8), we get

$$0=u(b)=u'(a)(b-a)+\int_{a}^{b}(b-s)v(s)ds\Rightarrow u'(a)=-\frac{1}{(b-a)}\int_{a}^{b}(b-s)v(s)ds$$

Now, this leads to

$$u(t)=\int_{a}^{t}(t-s)v(s)ds-\frac{t-a}{(b-a)}\int_{a}^{b}(b-s)v(s)ds:=\int_{a}^{b}g_0(t,s)v(s)ds \tag{4.9}$$

where

$$g_0(t,s)=\begin{cases}(t-s)-\dfrac{(t-a)(b-s)}{(b-a)} & a\le s\le t\\ -\dfrac{(t-a)(b-s)}{(b-a)} & t<s\le b,\end{cases}$$

as required in the Lemma.

Remark 4.4. As noted earlier in Remark 4.2, $g_0(s,t)$ is independent of $v(t)$ (cf (4.7). In the sense of Definition (4.2), the integral operator $\boldsymbol{G_0}:C[a,b]\to C[a,b]$ associated with $g_0(s,t)$ is given by

$$\boldsymbol{G_0}(v)=\int_{a}^{b}g_0(.,s)v(s)ds,\qquad \forall v\in C[a,b]. \tag{4.10}$$

Thus, the solution of (4.7) can be expressed as (cf (4.8))

$$u=\boldsymbol{G_0}(v). \tag{4.11}$$

Remark 4.5. Let $u \in C^2[a,b]$ such that $u(a)=u(b)=0$. and let $v \in C[a,b]$. As a first remark on "*BVP versus Integral Equation*", note that the function $w = u$ satisfies the BVP $w''(t)=v(t)$ subject to $w(a)=w(b)=0$ if and only if the solution of the integral equation (cf 4.11)

$$\boldsymbol{G_0}(f)=u \tag{4.12}$$

is given by $f = v$.

Remark 4.6. Some properties of $g_0(s,t)$ are as follows:

i. $g_0(t,a) = g_0(t,b) = 0$.

ii. $$\frac{\partial g_0}{\partial t}(t,s)=\begin{cases} 1-\dfrac{(b-s)}{(b-a)} & a \le s \le t \\ -\dfrac{(b-s)}{(b-a)} & t < s \le b \end{cases}$$

iii. $\lim_{s\to t^-} g_0(s,t) - \lim_{s\to t^+} g_0(s,t) = -1$, i.e., $g_0(s,t)$ has a jump discontinuity along the line. This is a typical property of the Green's function associated with 2nd order operators (cf (4.2)).

We set

$$\boldsymbol{G_1}(v) := \int_a^b g_1(.,s)v(s)ds \tag{4.13}$$

where $g_1(t,s)=\dfrac{\partial g_0(t,s)}{\partial t}$.

Next lemma describes a relationship between the solutions of BVP (4.1) and appropriate integral equation.

Lemma 4.B. (*BVP versus Integral Equation*) Suppose that BVP (cf (4.1))

$$\begin{cases} w'' = P(t)w' + Q(t)w + R(t), & a < t < b, \\ w(a)=0, \qquad w(b)=0. \end{cases} \tag{4.14}$$

has a unique solution. Let $u \in C^2[a,b]$ such that $u(a)=u(b)=0$. Consider the integral operator (cf (4.10) , (4.13))

$$\boldsymbol{G} := P\boldsymbol{G_1} + Q\boldsymbol{G_0} \tag{4.15}$$

Then $w = u$ is the solution of the BVP if and only if the integral equation

$$(\boldsymbol{I} - \boldsymbol{G})(v) = R \tag{4.16}$$

has a solution for $v = u''$.

Proof. Let $u \in C^2[a,b]$ such that $u(a) = u(b) = 0$. Consider $g_0(s,t)$, the green function associated with the linear operator $L_0(w) = w''$ where $w(a) = w(b)$ and the integral equation $\boldsymbol{G_0}(v) = u(t)$, $v \in C[a,b]$ (cf (4.11)).

Set $g_1(s,t) = \dfrac{\partial g_0(s,t)}{\partial t}$. Then differentiating both sides of (4.9) with respect to t, we get $\int_a^b g_1(.,s)v(s)ds = u'(t)$. Then in the notation of (4.13')

$$\boldsymbol{G_1}(v) = u'(t) \tag{4.17}$$

From (4.7), (4.11) and (4.17) we have (cf (4.1))

$$R(t) = u'' - P(t)u' - Q(t)u = v(t) - P(t)\boldsymbol{G_1}(v(t)) - Q(t)\boldsymbol{G_0}(v(t)) := v(t) - \boldsymbol{G}(v(t))$$

where $\boldsymbol{G}: C[a,b] \to C[a,b]$ is defined as

$$\boldsymbol{G}(v(t)) := (P(t)\boldsymbol{G_1}(v(t)) + Q(t)\boldsymbol{G_0}(v(t))). \tag{4.18}$$

Note that for a given R, we can determine $u(t)$, the solution of the BVP (4.14). In other words, $R(t)$ determines $v(t) := u''(t)$ which is the solution of the integral equation (4.11) (cf Remark 4.5) and consequently, it is the solution of the integral equation (4.12). Thus, $v(t) := u''(t)$ is the solution of the integral equation.

Hence, for a given R, $w = u$ is the solution of BVP (4.1) if and only if $v := u''$ is the solution of the integral equation

$$R = (\boldsymbol{I} - \boldsymbol{G})v. \tag{4.19}$$

Remark 4.7. The operator $\boldsymbol{G}$ (cf (4.18)) is a well-defined Fredholm operator of 2^{nd} kind which depends on $P(t)$ and $Q(t)$, the coefficients of the differential operator. Lemma 4.B describes that solving BVP (4.1) is equivalent to solving integral equation (4.16).

4.3 Collocation problem for a 2^{nd} order BVP

In this section we explain a collocation method for approximating the solution of BVP (4.1) which is based on piecewise cubic polynomials. There are several collocation methods. We discuss one such method. For this, assume that

$\eta_n:\ a = t_1 < t_2 < \cdots < t_{n+1} = b$

be a partition of interval [*a*,*b*] and let $\pi(3,\eta_n)$ denote the class of piecewise polynomials of degree 3 between the knots of η_n. Set

$$P_3X_n = \left\{ p \in C^2[a,b] \cap \pi(3,\eta_n) : p(a) = p(b) = 0 \right\}. \tag{4.20}$$

We consider the Collocation Equations based on the respective differential operator *L* (cf 4.2))

$$Lx_n(t_i) = R(t_i), \quad i = 1,2,...,n+1 \tag{4.21}$$

where $x_n \in P_3X_n$. We are interested in determining the parameters of x_n, the approximate solution of the BVP by solving the set of equations in (4.21). This process of finding x_n is referred to as a Collocation Problem for BVP (4.1) relevant to P_3X_n.

Definition. If there exists an $x_n \in P_3X_n$ that satisfies collocation equation (4.21), then $x_n(t)$ is called the approximate collocation solution of BVP (4.1) from P_3X_n.

Remark 4.8. (*Necessary conditions required to determine the parameters of* x_n) Note that each $x_n \in P_3X_n$ contains 4*n* parameters since

$x_n(t) = a_{3,i}t^3 + a_{2,i}t^2 + a_{1,i}t + a_{0,i}$.

on $[x_i x_{i+1}]$, $i = 1,2,...,n$. On the other hand,

i. the linear system (4.21) consists of $n + 1$ equations.

ii. there are two equations related to boundary conditions:

$$\left.\begin{array}{l} x_n(a) = 0 \\ x_n(b) = 0 \end{array}\right\}. \tag{4.22}$$

iii. because of continuity of the derivatives x_n (*t*) (cf 4.20), there are $3n - 3$ equations consisting of the constraints:

$$\lim_{t \to t_k^-} x_n^{(j)} = \lim_{t \to t_k^+} x_n^{(j)}, \qquad k = 2,3,...,n; \quad j = 0,1,2\,. \tag{4.23}$$

Therefore, the number of parameters "$4n$" in system (4.21) equals the number of equations "$(n+1) + 2 + 3(n-1)$".

4.4 Existence of Collocation solution

There are two approaches to discuss the existence of collocation solution by analyzing either the coefficient matrix of the concerned system of linear equations[3] or the integral equation of the form (4.19). We shall follow the latter approach. First, note that $v_n(t) = u_n''(t)$ is a piecewise linear polynomial between the knots of η_n if $u_n \in P_3 X_n$.

Definition 4.3. Let $\boldsymbol{P}_n$ be an operator on $C[a,b]$ that associates to each $h \in C[a,b]$ a unique piecewise linear polynomial l that interpolates h at t_i, $i = 1,2,\ldots,n+1$, the knots of η_n. We shall refer to $\boldsymbol{P}_n$ as η_n-l operator.

Remark 4.9. (i) $\boldsymbol{P}_n h = l$ is uniquely determined by the system of linear equations

$$\begin{cases} l(t_i) = h(t_i), & i = 1,2,...,n+1, \\ \lim_{t \to t_i^-} l(t) = \lim_{t \to t_i^+} l(t), & i = 2,...,n-1. \end{cases} \tag{4.24}$$

(ii) Two function $f, h \in C[a,b]$ have the same interpolating piecewise linear polynomial between the knots of η_n, i.e., $\boldsymbol{P}_n f = \boldsymbol{P}_n h$ if and only if $f(t_i) = g(t_i),\ i = 1,2,...,n+1$.

(iii) $\boldsymbol{P}_n$ is a projection on $C[a,b]$.

Lemma 4.C. (*Approximate solutions of BVP and Integral Equations*) Consider the η_n-l operator $\boldsymbol{P}_n$ and the integral operator G as defined above. Then solving the system of

[3] We note that the equations given in (4.15), (4.16) and (4.17) lead to a $4n \times 4n$ system of linear equations. Thus, the unique solution $x_n \in P_3 X_n$ will exist if $\boldsymbol{C}_{4n}$, the coefficient matrix of the system is invertible. In fact, a direct study of $\boldsymbol{C}_{4n}$ entails the use of some classical results concerning nonnegative matrices, diagonal dominance and Gershgorin theorems [67], [12]. Such a study suffers limitations and does not provide the most general results for collocation methods. The existence and uniqueness of the solution for problem (4.2) without getting into analysis of the coefficient matrix appear in the work of some Russian mathematicians: Vainikko [65], Karpelovskaya [35] and Shindler [58]. Later, their approach was followed by Russell et al who used piecewise polynomials as basic functions in the collocation method. Green's functions have a significant role in this approach. For this reason, integral equation method is preferred in order to determine the existence and uniqueness of collocation solution.

collocation equations $Lx_n(t_i) = R(t_i), \;\; i = 1,2,...,n+1$ (cf (4.21)) in $P_3 X_n$ is equivalent to solving the integral equation

$$\boldsymbol{P}_n R = v_n - \boldsymbol{P}_n \boldsymbol{G} v_n , \tag{4.25}$$

for $v_n = u_n''$.

Proof. As mentioned above, the integral operator $\boldsymbol{G}$ in (4.15) is exclusively based on the coefficients of the integral operator related to the BVP (4.1) and $g_0(s,t)$, the Green's function corresponding to the 2nd order differential operator $L_0 : X \to C[a,b]$ defined as $L_0(u) = u''$ for $u \in X = \{f \in C^2[a,b] : f(a) = f(b) = 0\}$.

By Lemma 4.B, we note that $x = u \in X$ is a unique solution of BVP $Lx = f$ (cf (4.2)) if and only if $u'' = v$ is a unique solution of the integral equation $(I - \boldsymbol{G})v = R$.

Using an argument given in the Lemma, we conclude that $x_n = u_n \in P_3 X_n$ is a solution of $Lx_n(t_i) = f(t_i), \; i = 1,2,...,n$ if and only if $v_n = u_n''$ is a solution of the linear system of $(I - \boldsymbol{G})v_n(t_i) = R(t_i), \; i = 1,2,...,n$.

Now consider the operator $\boldsymbol{P}_n$ that assigns a unique piecewise linear Lagrange polynomial to $f \in C[a,b]$ at the n+1 knots t_i, $i = 1,2,...,n+1$. Then for $f, g \in C[a,b]$, $f(t_i) = g(t_i), \; i = 1,2,...,n+1$, if and only if $\boldsymbol{P}_n f = \boldsymbol{P}_n h$. Thus, $v_n = u_n''$ is a unique solution of $(I - \boldsymbol{G})v_n(t_i) = R(t_i), \; i = 1,2,...,n+1$ if and only if $v_n = u_n''$ is a unique solution of

$$\boldsymbol{P}_n (I - \boldsymbol{G}) v_n = \boldsymbol{P}_n R . \tag{4.26}$$

Note that v_n is a piecewise linear polynomial between the knots of η_n. By Remark 4.9 (iii) $\boldsymbol{P}_n v_n = v_n$. Because of this, equation (4.26) reduces to $\boldsymbol{P}_n R = (\boldsymbol{I} - \boldsymbol{P}_n \boldsymbol{G}) v_n$. Summarizing the above discussion, we conclude that $x_n = u_n \in P_3 X_n$ is a solution of $Lx_n(t_i) = f(t_i), \; i = 0,1,...,n$ if and only if $v_n = u_n''$ is a unique solution of $\boldsymbol{P}_n R = (\boldsymbol{I} - \boldsymbol{P}_n \boldsymbol{G}) v_n$.

Remark 4.10. (*Condition for existence of unique collocation solution*) From the above discussion it follows that collocation equation (4.21) has a unique solution if and only the operator $(\boldsymbol{I}-\boldsymbol{P}_n\boldsymbol{G})$ is invertible. The following theorem provides the condition(s) that guarantee the non-singularity of $(\boldsymbol{I}-\boldsymbol{P}_n\boldsymbol{G})$.

Theorem 4.B. [48, Theorem 8.2] If BVP (4.1) has a unique solution, then there is a positive integer N_0 such that $(\boldsymbol{I}-\boldsymbol{P}_n\boldsymbol{G})^{-1}$ exists for all $n > N_0$. Moreover,

$\left\|(\boldsymbol{I}-\boldsymbol{P}_n\boldsymbol{G})^{-1}u\right\| \le \gamma \left\|u\right\|, \forall n > N_0$ and $\forall u \in C[a,b]$,

where constant γ is independent of both u and N_0. In particular, sequence $\left\{(\boldsymbol{I}-\boldsymbol{P}_n\boldsymbol{G})^{-1}\right\}_{n\ge N_0}$ is uniformly bounded.

Proof of Theorem 4.B is based on some Lemmas narrated below. The integral operator $\boldsymbol{I}-\boldsymbol{G}$ which appears in (4.16)) may be regarded as a linear operator on the space V = $C[a,b]$ endowed with Chebyshev norm. Also, η_n will be regarded as a uniform partition of interval $[a,b]$, i.e., with $h = (b - a)/n$, we shall consider $t_i = a + (i - 1)h$, $i = 1,2,\ldots,n+1$

Lemma A. [48, Lemma 1, page 292] The sequence of projections $\boldsymbol{P}_n$, n = 1,2,…, converges strongly to identity and is uniformly bounded.

Lemma B. [48, Lemma 2, page 293] $\left\|\boldsymbol{G}-\boldsymbol{P}_n\boldsymbol{G}\right\| \to 0$ as $n \to \infty$.

Lemma C. [48, Neumann's Theorem, page 294] Let $S,T,T^{-1} \in B[V]$ and let $\Delta = \left\|T^{-1}\right\|\left\|S-T\right\|$. If $\Delta < 1$, then

i. S^{-1} exists

ii. $S^{-1} \in B[V]$

iii. $\left\|S^{-1}-T^{-1}\right\| \le \dfrac{\left\|T^{-1}\right\|\left\|T-S\right\|}{1-\Delta}$.

Finally, we provide

Proof of Theorem 4.B [48, page 294]. First note that the BVP (4.1) has a unique solution. Therefore, the integral equation $R = (\boldsymbol{I} - \boldsymbol{G})v$ (cf (4.16)) will also have a unique solution which means that $(\boldsymbol{I} - \boldsymbol{G})^{-1}$ exists. Moreover, $(\boldsymbol{I} - \boldsymbol{G})^{-1}$ is bounded since $\boldsymbol{G}$ is compact. Now let $T = \boldsymbol{I} - \boldsymbol{G}$ and $S = \boldsymbol{I} - \boldsymbol{P}_n\boldsymbol{G}$ in Lemma C. Then (cf Lemma B) $\|S - T\| = \|\boldsymbol{G} - \boldsymbol{P}_n\boldsymbol{G}\| \to 0$ as $n \to 0$, i.e., $\exists N_0$ so that $\Delta = \|T^{-1}\|\|S - T\| = \|(\boldsymbol{I} - \boldsymbol{G})^{-1}\|\|\boldsymbol{G} - \boldsymbol{P}_n\boldsymbol{G}\| < 1$ for $n > N_0$. From Lemma C (i) it follows that $(\boldsymbol{I} - \boldsymbol{P}_n\boldsymbol{G})^{-1}$ exists for $n > N_0$. Furthermore, the sequence $\left\{(\boldsymbol{I} - \boldsymbol{P}_n\boldsymbol{G})^{-1}\right\}_{n>N_0}$ is uniformly bounded.

5. Orthogonal Collocation Points

It is known that the Finite Element Method usually provides order h^4 accuracy [30] when applied to problem (2.5). On the other hand, the order of accuracy due to CM is h^2 in general. Recall the variation in the accuracy of approximate solutions obtained in Example 3.1. This variation is due to the selection of collocation points and the basic functions. While going through Tables 3.1-3.3, we note that the collocation points emerging as zeros of certain orthogonal polynomials produced promising results even better to finite difference method (cf Section 3.4). These observations motivate us to study the structure of this type of points carefully.

5.1 Gaussian collocation points

The most commonly used points in the method of orthogonal collocation are the "**Gaussian points**". These are, in fact, the zeros of Legendre polynomials when shifted to a given interval by a linear transformation. The construction of these points is carried out as follows

i. The 2nd degree Legendre polynomial has two zeros: $t_1 = -\frac{1}{\sqrt{3}}, \; t_2 = \frac{1}{\sqrt{3}}$

ii. For any interval $[x_i, x_{i+1}]$, set

$$z_{i,1} = x_i + \frac{h_i}{2}(1+t_1). \qquad z_{i,2} = x_i + \frac{h_i}{2}(1+t_2)$$

where $h_i = x_{i+1} - x_i$

iii. The points $z_{i,1}, z_{i,2}$ are termed as a pair of Gaussian points in the interval $[x_i, x_{i+1}]$ and retain the orthogonality property of Legendre polynomials, i.e.,

$$\int_{x_i}^{x_{i+1}} p(t) p_{i,2}(t) dt = 0, \qquad \forall p \in \pi_1$$

when $p_{i,2}(t) = (t - z_{i,1})(t - z_{i,2})$.

Here, the notation π_k stands for the class of all polynomials up to degree *k*.

5.2 Zeros of orthogonal 0-interpolants

In order to define orthogonal zero-interpolants on an interval [a,b], we fix a finite number of zeros, say, $t_1, t_2, \ldots, t_m$ in [a,b] and set $\varphi_1(t) = \prod_{i=1}^{m}(t-t_i)^{n_i}$. The function φ_1 interpolates the 0-function at the multiple nodes $t_1, t_2, \ldots, t_m$ in the sense of Hermite. Next we define a sequence of functions $\varphi_j \in \pi_{M+j}, j = 2,3,\ldots,l$ with $M = \left(\sum_{i=1}^{m} n_i\right) - 1$ in such a way that the functions $\varphi_j \in \pi_{M+j}$, $j = 1,2,\ldots,l$ are mutually orthogonal on [a,b]. We shall call the functions φ_j "Orthogonal zero-interpolants (*OZI*)". Some properties of φ_j are as follows:

i. $\varphi_j = p_j \varphi_1$ for some $p_j \in \pi_j$.

ii. The factor polynomials p_j, $j = 0,1,2,\ldots,l$ are orthogonal with respect to the weight function $w = \varphi_1^2$ on [a,b].

iii. The function φ_1 is a factor of all φ_j's. Therefore, the multiple zeros of φ_1 are also the zeros of each φ_j.

iv. φ_j has j distinct real zeros different from $t_i, i = 1,2,\ldots,m$, which lie inside the open interval (a,b).

We shall use shifted zeros of certain *OZI*'s in our proposed method for approximating the solution of BVP's. Structure of these zeros for the 2nd degree *OZI* is explained below.

5.3 Orthogonal pair of points

i. Fix a point a_1 in [– 1, 1] and set $\varphi_1(t) = t - a_1$.

ii. Set $\varphi_2(t) = (t - \alpha_1)\varphi_1(t)$ where

$$a_2 = \alpha_1 = \frac{\langle t\varphi_1, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle}. \tag{5.1}$$

Here, the notation $\langle .\,, . \rangle$ stands for inner product and is defined as $\langle h, g \rangle := \int_{-1}^{1} h(t)g(t)dt$.

iii. a_1 and a_2 form a pair of zeros lying in [– 1, 1] for 2nd degree *OZI* φ_2.

Definition 5.1. We shall call (a_1, a_2) an orthogonal pair of points in [– 1,1] if $a_1 \in [-1,1]$ and a_2 is given by (5.1).

Definition 5.2. Let (a_1, a_2) be an orthogonal pair of points in [– 1,1]. For an interval $[x_i, x_{i+1}]$, set

$$z_{i,1} = x_i + \frac{h_i}{2}(1+a_1). \quad z_{i,2} = x_i + \frac{h_i}{2}(1+a_2) \tag{5.2}$$

where $h_i = x_{i+1} - x_i$. The pair $(z_{i,1}, z_{i,2})$ will be termed as the "shifted orthogonal pair of (a_1, a_2)" in the interval $[x_i, x_{i+1}]$.

Remark 5.1. Let (a_1, a_2) be an orthogonal pair in [– 1, 1] and let $(z_{i,1}, z_{i,2})$ be its shifted orthogonal pair in $[x_i, x_{i+1}]$. Then

i) in general, $\int_{x_i}^{x_{i+1}} (t - z_{i,1})^2 (t - z_{i,2}) dt = 0$.

ii) if we let $a_1 = \frac{-1}{\sqrt{3}}$ then $a_2 = \frac{1}{\sqrt{3}}$, i.e., (a_1, a_2) is a pair of the zeros of 2nd degree Legendre polynomial. Moreover,

$$\int_{x_i}^{x_{i+1}} (t - z_{i,1})(t - z_{i,2}) dt = 0, \tag{5.3}$$

since $(z_{i,1}, z_{i,2})$ is a shifted pair of $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, the zeros of 2nd degree Legendre polynomial, in $[x_i, x_{i+1}]$.

5.4 Restriction on the selection of 1st zero

Is there any restriction on the choice of first knot in an interval $[x_i, x_{i+1}]$? Our answer to this question is affirmative. If we require the second knot to be distinct in $[x_i, x_{i+1}]$, we cannot choose the mid-point of the interval as the first knot. Otherwise, first and second knot will be identical as we see below in case of the interval [–1,1]:

Select $a_1 = 0$ in $[-1,1]$, then (cf (5.1)) $a_2 = \alpha_1 = \frac{\langle t\varphi_1, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle}$ with $\varphi_1(t) = t - a_1$. Since $\langle t\varphi_1, \varphi_1 \rangle = \int_{-1}^{1} t^3 dt$, we conclude that $a_2 = 0$ which is undesirable.

Remark 5.2. We shall be in need of two distinct orthogonal collocation points lying in each subinterval while approximating the solution of two-point BVP by the OCM.

6. Piecewise Hermite Polynomials

This chapter deals with specific piecewise polynomials which form a basis of a finite dimensional approximating subspace of *X*, the solution space of BVP (4.1). These polynomials will be used in the construction of an approximate solution of a given BVP. Recall that Hermite interpolating polynomial interpolates a given function *f* and its finite number of consecutive derivatives at pre-assigned points. Here, we explain basic structure of piecewise cubic Hermite interpolates[4].

6.1 Cubic Hermite polynomials

We start with the general definition of cubic Hermite polynomial and discuss its fundamental polynomials.

Definition 6.1 [49]. The cubic Hermite polynomial (C-H polynomial)for a function $f:[t_1,t_2]\to\Re$ is a unique 3rd degree polynomial *p* which satisfies the following constraints at a pair of given knots t_1 and t_2:

$$p^{(j)}(t_i)=f^{(j)}(t_i),\qquad i=1,2;\ j=0,1. \tag{6.1}$$

The four constraints imposed on the structure of a C-H polynomial give rise to four fundamental polynomials which guarantee its existence and are defined below.

Definition 6.2. A cubic Hermite polynomial of a given function and a given pair of knots t_1 and t_2 can be expressed in terms of four Fundamental Polynomials as follows:

$$p(x)=f(t_1)\phi_1(x)+f(t_2)\phi_2(x)+f'(t_1)\psi_1(x)+f'(t_2)\psi_2(x) \tag{6.2}$$

where the fundamental polynomials $\phi_i,\psi_i\in\pi_3,\ i=1,2$, satisfy the conditions

$$\begin{cases}\phi_i(t_j)=\delta_{ij} & \psi_i(t_j)=0\\ \phi_i'(t_j)=0 & \psi_i'(t_j)=\delta_{ij}\end{cases},\quad i,j=1,2. \tag{63}$$

[4] For the structure of piecewise quartic Hermite interpolates, see Section 11.2

The polynomials $\phi_1, \phi_2, \psi_1, \psi_2$ and their first derivatives assumes the value either 1 or 0 at the prescribed knots t_1 and t_2. In accordance with their graphical appearance, we tag them respectively by $\phi_{L,i}, \phi_{R,i}, \psi_{U,i}, \psi_{L,i}$. We introduce the following terminology to express them in a simpler way:

Definition 6.3. We shall say that a polynomial q_i satisfies the conditions (a,b,c,d) at a given pair of points t_i and t_{i+1} if $q_i(t_i) = a,\ q_i'(t_i) = b,\ q_i(t_{i+1}) = c,\ q_i'(t_{i+1}) = d$

Fig 6.1: $\phi_{L,i}$

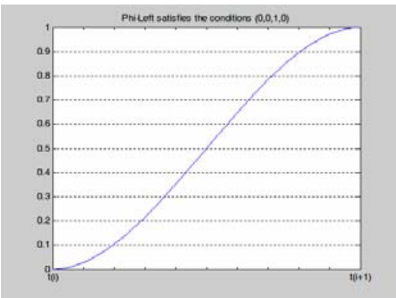


Fig 6.2: $\phi_{R,i}$

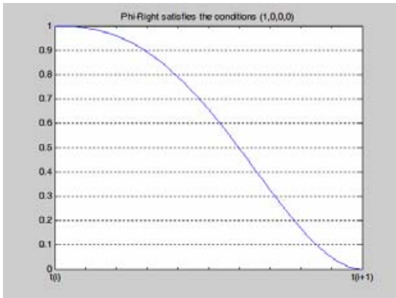


Fig 6.3: $\psi_{L,i}$

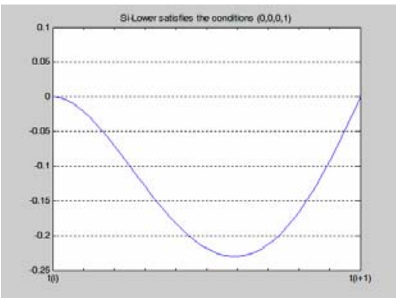
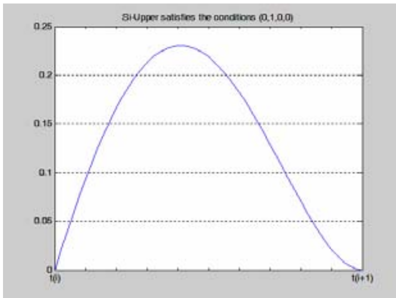


Fig 6.4: $\psi_{U,i}$



Using the above definition, we describe the structure of the fundamental polynomials as follows:

$$\left.\begin{array}{lll} (1)\ (\phi\text{-Left}): & \phi_1 = \phi_{L,i} & \text{satisfies } (0,0,1,0) \text{ conditions} \\ (2)\ (\phi\text{-Right}): & \phi_2 = \phi_{R,i} & \text{satisfies } (1,0,0,0) \text{ conditions} \\ (3)\ (\psi\text{-Upper}): & \psi_1 = \psi_{U,i} & \text{satisfies } (0,1,0,0) \text{ conditions} \\ (4)\ (\psi\text{-Lower}): & \psi_2 = \psi_{L,i} & \text{satisfies } (0,0,0,1) \text{ conditions} \end{array}\right\} \quad (6.2)$$

6.2 Explicit representation of fundamental C-H polynomials

Using the four interpolation conditions from (6.2), an explicit representation of each fundamental C-H polynomial for the knots x_1 and x_2 is given below.

i. ϕ - Left: $\phi_{L,i}$

$$\phi_{L,i}(t)=\begin{cases}\frac{1}{h^3}(t-x_i)^3+B_L(t-x_i)^2(t-x_{i+1}), & t\in[x_i,x_{i+1}]\\ 0, & \text{else where}\end{cases} \tag{6.3}$$

with $B_L=\frac{-3}{h^3}$

ii. ϕ -Right: $\phi_{R,i}$

$$\phi_{R,i}(t)=\begin{cases}\frac{-1}{h^3}(t-x_{i+1})^3+B_R(t-x_i)(t-x_{i+1})^2, & t\in[x_i,x_{i+1}]\\ 0, & \text{else where}\end{cases} \tag{6.4}$$

with $B_R=\frac{3}{h^3}$

iii. ψ -Upper: $\psi_{U,i}$

$$\psi_{U,i}(t)=\begin{cases}C_U(t-x_i)(t-x_{i+1})^2, & t\in[x_i,x_{i+1}]\\ 0, & \text{else where}\end{cases} \tag{6.5}$$

with $C_U=\frac{1}{h^2}$

iv. ψ -Lower: $\psi_{L,i}$

$$\psi_{L,i}(t)=\begin{cases}C_L(t-x_i)^2(t-x_{i+1}) & t\in[x_i,x_{i+1}]\\ 0, & \text{else where}\end{cases} \tag{6.6}$$

with $C_L=\frac{1}{h^2}$.

Next, we explain the structure of piecewise cubic Hermite polynomials on a set of uniformly distributed knots.

6.3 Piecewise C-H polynomials

A set of real numbers x_i's with $x_1<x_2<\ldots<x_{n+1}$ where the behavior of an approximating function g is constrained is called the set of knots of g. We say that the knots are uniformly distributed if $x_{i+1}-x_i$ is constant for $i=1,2,\ldots,n$,

For our further work we shall consider uniformly distributed knots on the interval $[a,b]$ which are given as

$$a = x_1 < x_2 < \ldots < x_{n+1} = b\,. \qquad (6.7)$$

Definition 6.4. A real-valued function $g:[a,b]\to\Re$ is called a piecewise cubic Hermite polynomial with knots at x_i's (cf (6.7)) if it is a linear combination of all fundamental C-H polynomials $\phi_{L,i}$, $\phi_{R,i}$, $\psi_{U,i}$, $\psi_{L,i}$ each with knots at x_i and x_{i+1}, $i = 1,2,\ldots,n$.

Definition 6.5. We say that $g:[a,b]\to\Re$ is a piecewise cubic Hermite interpolate of a given function $h:[a,b]\to\Re$ at the knots x_i's (cf (6.7)) if

$$g^{(j)}(x_i) = f^{(j)}(x_i),\ i = 1,2,\ldots,n+1;\ j = 0,1\,.$$

7. Orthogonal Collocation Methods

Collocation method was described in Section 3.1 along with its application to a simple BVP (see Example 3.1). There are different ways to implement this method by considering various types of collocation points and the functions forming a basis of finite dimensional approximating space. While considering a finite dimensional approximating space, we shall construct its basis where each member consists of at most two pieces of fundamental C-H polynomials (cf Chapter 6). Here, we are interested to use collocation points which are related to certain orthogonal polynomials (cf Chapter 5). A collocation method based on such points is known as "Orthogonal Collocation Method (OCM)". Here, two Legendre zeros (cf Section 5.1) or two *OZI* points (cf Definitions 5.1, 5.2) will be coupled with basis of approximating subspace. With this understanding we provide a complete description of the OCM-based algorithm for finding approximate solution of the two-point BVP with homogeneous boundary conditions (cf (2.2)):

$$\begin{cases} w'' = P(t)w' + Q(t)w + R(t), \ a < t < b, \\ w(a) = 0, \qquad w(b) = 0. \end{cases} \tag{7.1}$$

Recall that the operator form of (7.1) is given by $L(\varsigma) = R$ (cf (2.5)) where

$$L(\varsigma) = \frac{d^2\varsigma}{dx^2} - P(x)\frac{d\varsigma}{dx} - Q(x)\varsigma, \qquad \varsigma \in X , \tag{7.2}$$

and X is the underlying solution space given by (cf (2.4))

$$X = \left\{ \varsigma \in C^2[a,b] : \varsigma(a) = \varsigma(b) = 0 \right\}. \tag{7.3}$$

7.1 Solution of BVP based on Piecewise C-H polynomials

For the implementation of *OCM* to BVP (7.1), we subdivide interval $[a,b]$ into n subintervals by uniformly distributed knots as given below (cf (6.7)):

$$D_n : a = x_1 < x_2 < \ldots < x_{n+1} = b . \tag{7.4}$$

Based on subdivision (7.4), we construct N = 2n piecewise C-H polynomials $\xi_1, \xi_2, \ldots, \xi_N$ which satisfy zero boundary conditions, and then consider the N-dimensional approximating subspace $X_N = \langle \phi_1, \phi_2, \ldots, \phi_N \rangle$ of X. The approximate solution of (2.2) will be of the form

$$\varsigma_N = c_1\phi_1 + c_2\phi_2 + \cdots + c_N\phi_N \tag{7.5}$$

where c_i's are unknowns. Below we explain the procedure of finding the parameters c_i's.

7.2 Procedure for computing an OCM based solution

Step 1: Using the representations of $\phi_{L,i}, \phi_{R,i}, \psi_{L,i}, \psi_{u,i}$ from (6.3)-(6.6), we define

$$\phi_i(t) = \begin{cases} \phi_{L,i}(t), & t \in [x_i, x_{i+1}], \\ \phi_{R,i+1}(t), & t \in [x_{i+1}, x_{i+2}], i = 1,2,\ldots n-1. \\ 0, & \text{elsewhere,} \end{cases} \tag{7.6}$$

and

$$\psi_i(t) = \begin{cases} \psi_{L,i}(t), & t \in [x_i, x_{i+1}], \\ \psi_{U,i+1}(t), & t \in [x_{i+1}, x_{i+2}], i = 1,2,\ldots n-1. \\ 0, & \text{elsewhere,} \end{cases} \tag{7.7}$$

Now define ξ_i's by means of the following ordered-sets equation:

$$\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \ldots, \xi_{2n-3}, \xi_{2n-2}, \xi_{2n-1}, \xi_{2n}\} = \{\psi_{U,1}, \psi_1, \phi_1, \psi_2, \phi_2, \ldots, \psi_{n-2}, \phi_{n-1}, \psi_{n-1}, \psi_{L,n}\}. \tag{7.8}$$

It may be noted that ξ_i's satisfy zero boundary conditions, i.e.,

$$\xi_i(a) = \xi_i(b) = 0, \;\; i = 1,2,\ldots,2n.$$

Also, note that there is a relation between the number of subintervals of $[a,b]$ and the number of basis polynomials ξ_i's , i.e.,

$$N = 2n. \tag{7.8}$$

Step 2: Since $N = 2n$, we require n pairs of *OZI* points $z_{i,1}, z_{i,2}$ in the interval $[x_i, x_{i+1}]$ with respect to the orthogonal pair (a_1,a_2) (same pair may be be considered for each subinterval $[x_i, x_{i+1}], i = 1,2,\ldots,n$) to solve $N \times N$ system of linear equations (cf (7.1), 7.2), (7.5)). This leads to the linear system

$$\left.\begin{array}{l} c_1 L\xi_1(z_{i,1}) + c_2 L\xi_2(z_{i,1}) + \cdots + c_N L\xi_N(z_{i,1}) = R(z_{i,1}) \\ c_1 L\xi_1(z_{i,2}) + c_2 L\xi_2(z_{i,2}) + \cdots + c_N L\xi_N(z_{i,2}) = R(z_{i,2}) \end{array}\right\} i = 1,2,\ldots,n \tag{7.9}$$

Step 3: The coefficient matrix of the system (7.9) is pent-diagonal and is given by

$$
A=\begin{bmatrix}
\tilde{\psi}_{U,1,(1)} & \tilde{\phi}_{L,1,(1)} & \tilde{\psi}_{L,1,(1)} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\tilde{\psi}_{U,1,(2)} & \tilde{\phi}_{L,1,(2)} & \tilde{\psi}_{L,1,(2)} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & \tilde{\phi}_{R,2,(1)} & \tilde{\psi}_{U,2,(1)} & \tilde{\phi}_{L,2,(1)} & \tilde{\psi}_{L,2,(1)} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \tilde{\phi}_{R,2,(2)} & \tilde{\psi}_{U,2,(2)} & \tilde{\phi}_{L,2,(2)} & \tilde{\psi}_{L,2,(2)} & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \tilde{\phi}_{R,3,(1)} & \tilde{\psi}_{U,3,(1)} & \tilde{\phi}_{L,3,(1)} & \tilde{\psi}_{L,3,(1)} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \tilde{\phi}_{R,3,(2)} & \tilde{\psi}_{U,3,(2)} & \tilde{\phi}_{L,3,(2)} & \tilde{\psi}_{L,3,(2)} & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \tilde{\phi}_{R,n-1,(1)} & \tilde{\psi}_{U,n-1,(1)} & \tilde{\phi}_{L,n-1,(1)} & \tilde{\psi}_{L,n-1,(1)} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \tilde{\phi}_{R,n-1,(2)} & \tilde{\psi}_{U,n-1,(2)} & \tilde{\phi}_{L,n-1,(2)} & \tilde{\psi}_{L,n-1,(2)} & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \tilde{\phi}_{R,n,(1)} & \tilde{\psi}_{U,n,(1)} & \tilde{\psi}_{L,n,(1)} \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \tilde{\phi}_{R,n,(2)} & \tilde{\psi}_{U,n,(2)} & \tilde{\psi}_{L,n,(2)}
\end{bmatrix} \tag{7.10}
$$

Significance of sign ~ on each piecewise polynomial in the above matrix is explained by the following examples:

$\tilde{\phi}_{R,k,(1)}$ stands for $L\phi_{R,k}(z_{k,1})$ and $\tilde{\psi}_{U,k,(2)}$ stands for $L\psi_{U,k,(2)}(z_{k,2})$. Note that k in these represents the k-th subinterval $[x_k, x_{k+1}]$ on which the piecewise polynomials like $\phi_{R,k}$ and $\psi_{U,k}$ are defined. Moreover, the $z_{k,1}, z_{k,2} \in [x_k, x_{k+1}]$.

Step 4: The solution of the system (7.9) can be determined by using an appropriate MATLAB command "*A**b*" or MATLAB function "bicgstab".

Remark 7.1. The MATLAB program [72] designed by Yousuf for penta-diagonal coefficient matrix has an advantage to the MATLAB equation solvers. Its description is given in the following sections

7.3 MATLAB program for Pent-diagonal system of linear equations [72]

% Function to solve a Penta Diagonal System of Equations:

%

% Inputs

% Main diagonal: d(i), i=1:N

% 1st Upper Diagonal: a_2(i), i=2:N

% 2nd Upper Diagonal: a_1(i), i=3:N

% 1st Lower Diagonal: c_1(i), i=1:N-1

```
% 2nd Lower Diagonal:         c2(i),  i=1:N-2
% Right side of the System:   RS(i),   i=1:N
%
% Output
% Solution of the system (Row vector form):            C = [c1, c2,…, cN]
%
                         function C = penta_diagonal(a1,a2,d,c1,c2,RS)
%
N = length(RS);
%
for i=2:N-1
        xmult   =a2(i)/d(i-1);
        d(i)      = d(i)-xmult*c1(i-1);
        c1(i)     = c1(i)-xmult*c2(i-1);
        R(i)      = RS(i)-xmult*RS(i-1);
        xmult   = a1(i+1)/d(i-1);
        a2(i+1) = a2(i+1)-xmult*c1(i-1);
        d(i+1)  = d(i+1)-xmult*c2(i-1);
        R(i+1)  = RS(i+1)-xmult*RS(i-1);
 end
 %
        xmult   = a2(N)/d(N-1);
        d(N)     = d(N)-xmult*c1(N-1);
        C(N)     = (RS(N)-xmult*RS(N-1))/d(N);
        C(N-1) = (RS(N-1)-c1(N-1)*C(N))/d(N-1);
 %
 for i=N-2:-1:1
   C(i)  = (RS(i)-c1(i)*C(i+1)-c2(i)*C(i+2))/d(i);
End
```

8. Error Analysis

In this chapter, we derive error estimate for the collocation method based on *OZI points* discussed in Section 5.3. For this, we follow a technique of Prenter and Russell [47], while considering the *Gaussian knots*. Their approach is based on the notion of Green's function (cf Definition 4.1) .

To begin with, we consider the differential operator $L : X \to X$ defined as

$$L(w) = \frac{d^2w}{dx^2} - P(x)\frac{dw}{dx} - Q(x)w, \qquad w \in X, \tag{8.1}$$

where $X = \{w \in C^2[a,b] : w(a) = w(b) = 0\}$ is the solution space of the BVP $L(w) = R$ which can also be expressed as

$$\begin{cases} w'' = P(t)w' + Q(t)w + R(t), & a < t < b, \\ w(a) = 0, \qquad w(b) = 0. \end{cases} \tag{8.2}$$

The problem (8.2) has a unique solution whenever $P(t)$, $Q(t)$ and $R(t)$ are real-valued continuous functions on $[a,b]$ with $Q(t) > 0$ for $a \le t \le b$. Our main result on error estimate is given in the next section.

8.1 Error estimate for OCM based on a orthogonal pair of points

Our main result requires the structure of the approximate solution by OCM relevant to a pair of orthogonal points in [–1 ,1]. We consider K_n, a uniform partition of [a ,b], given by:

$$K_n:\ a = x_1 < x_2 < \cdots < x_{n+1} = b \tag{8.3}$$

where $x_i = a + (i-1)h, i = 1,2,...,n+1$ with $h = (b - a) / n$.

Assume that $\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \ldots, \xi_{2n-3}, \xi_{2n-2}, \xi_{2n-1}, \xi_{2n}\}$ is the set of piecewise Hermite cubic polynomials (cf (7.8)). Then $X_N = \langle \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \ldots, \xi_{2n-3}, \xi_{2n-2}, \xi_{2n-1}, \xi_{2n} \rangle$ with $N = 2n$ is an N-dimensional space. Note that each function $\zeta \in X_N$ satisfies the Dirichlet boundary conditions at a and b. Moreover, ζ'' is piecewise continuous on [a,b] with discontinuities occurring at the partitioning points of K_n given by (8.3). We fix $a_1 \in [-1,1]$

and determine a_2 which along with a_1 provides an orthogonal pair of points in [–1,1] (cf Definition 5.1). Following Definition 5.2, we consider two *OZI* points "$z_{i,1}$ and $z_{i,2}$" in each interval $[x_i, x_{i+1}]$, $i = 1,2,\ldots,n$, with respect to the orthogonal pair (a_1,a_2). We follow the procedure of Section 7.2 to construct the approximate solution $\varsigma_{N,a_1,a_2} \in X_N$ for the BVP (8.2). Our main result is as follows:

Theorem 8.1. *Let w be the unique solution of BVP (8.2). Let $R''(t)$ is piecewise continuous on [a ,b] such that its discontinuities occur at most at the partitioning points x_i, i = 1,2,…,n of the interval [a,b] as given in (8.3). Let $h = x_{i+1} - x_i$, $i = 1,2,\ldots n$. If $\varsigma_{N,a_1,a_2} \in X_N$ is the approximate solution of (8.2) based on the orthogonal pair (a_1,a_2) in [–1,1] as defined above, then*

$$\lim_{a_1 \to \frac{-1}{\sqrt{3}}} \left\| w - \varsigma_{N,a_1,a_2} \right\|_\infty = O(h^{3.5}). \qquad (8.4)$$

8.2 Some preliminaries

Our proposed proof of Theorem 8.1 depends on the notion of Green's function and properties of the orthogonal pair (a_1,a_2).

Lemma 8.1. *Let $a_1 \in \Re$. Then the orthognality condition $\int_{-1}^{1} (t - a_1)^2 (t - a_2) dt = 0$ implies that $a_2 = \dfrac{-2a_1}{1+3a_1^2}$. In addition,*

(i) $a_2 \in \left[0, \dfrac{1}{\sqrt{3}}\right]$ *if* $a_1 \le 0$,

(ii) $a_2 \in \left[\dfrac{1}{2\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$ *if* $a_1 \in \left[-1, \dfrac{-1}{\sqrt{3}}\right] \cup \left[\dfrac{-1}{\sqrt{3}}, 0\right]$,

(iii) $a_2 \to \dfrac{1}{\sqrt{3}}$ *if* $a_1 \to \dfrac{-1}{\sqrt{3}}$.

Proof. Note that $0 = \int_{-1}^{1} (t - a_1)^2 (t - a_2) dt = -\dfrac{2}{3}(2a_1 + a_2) - 2a_1^2 a_2$ **.** Therefore, $a_2 = \dfrac{-2a_1}{1+3a_1^2}$.

For (i) $a_1 \le 0 \Rightarrow a_2 = \dfrac{-2a_1}{1+3a_1^2} \ge 0$. On the other hand assume that $a_2 > \dfrac{1}{\sqrt{3}}$ i.e., $\dfrac{-2a_1}{1+3a_1^2} > \dfrac{1}{\sqrt{3}}$. Then $0 > 3a_1^2 + 2\sqrt{3}a_1 + 1 = (\sqrt{3}a_1 + 1)^2$ which is not possible. Thus, $a_2 \in \left[0, \dfrac{1}{\sqrt{3}}\right]$.

Next, if $a_1 \in \left[-1, \dfrac{-1}{\sqrt{3}}\right]$ then a simple manipulation leads to $\dfrac{1}{2\sqrt{3}} \le \dfrac{-2a_1}{1+3a_1^2} \le 1$. From (i), we note that $a_2 \le \dfrac{1}{\sqrt{3}}$. Therefore, $a_2 \in \left[\dfrac{1}{2\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$. Similarly, if $a_1 \in \left[\dfrac{-1}{\sqrt{3}}, 0\right]$ then again we find that $a_2 \in \left[\dfrac{1}{2\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$.

Proof of assertion (iii) follows from the following observation:

$$\lim_{a_1 \to \frac{-1}{\sqrt{3}}} a_2 = \lim_{a_1 \to \frac{-1}{\sqrt{3}}} \frac{-2a_1}{1+3a_1^2} = \frac{1}{\sqrt{3}}.$$

Lemma 8.2. Let (a_1,a_2) be an orthogonal pair of points in [– 1, 1] and let $(z_{i,1}, z_{i,2})$ be its shifted pair in the interval $[x_i, x_{i+1}]$. If we set $\varphi_{2,a_1,a_2}(t) := (t - z_{i,1})(t - z_{i,2})$, then

$$\lim_{a_1 \to \frac{-1}{\sqrt{3}}} \int_{x_i}^{x_{i+1}} \varphi_{2,a_1,a_2}(t)dt = 0.$$

Proof. Let $b_1 = \dfrac{-1}{\sqrt{3}}$ and $b_2 = \dfrac{-2b_1}{1+3b_1^2}$. Then by Lemma 8.1, (b_1,b_2) is a pair of the zeros of 2nd degree Legendre polynomial. Let $\left(z_{i,1}^*, z_{i,2}^*\right)$ be its shifted pair in $[x_i, x_{i+1}]$. Set $\varphi_{2,b_1,b_2}(t) := (t - z_{i,1}^*)(t - z_{i,2}^*)$. Then

$$\lim_{a_1 \to \frac{-1}{\sqrt{3}}} \int_{x_i}^{x_{i+1}} \varphi_{2,a_1,a_2}(t)dt = \int_{x_i}^{x_{i+1}} \varphi_{2,b_1,b_2}(t)dt. \tag{8.5}$$

The right hand side of (8.5) vanishes due to Remark 5.1 (ii). This completes the proof.

8.3 Proof of main result

In order to prove Theorem 8.1, assume that $w \in X$ is a unique solution of (8.2) and that its OCM based approximate solution $\varsigma_{N,a_1,a_2} \in X_N$ (cf (7.5)) is of the form

$$\varsigma_{N,a_1,a_2} = c_1\xi_1 + c_2\xi_2 + \cdots + c_N \xi_N \ . \qquad (8.6)$$

Since the solution of the BVP can be expressed in terms of Green's function, say $G(s,t)$ of its differential operator L, by (4.3), we have

$$\begin{cases} w(s) = \int_a^b G(s,t) Lw(t)dt \\ \varsigma_{N,a_1,a_2}(s) = \int_a^b G(s,t) L\varsigma_{N,a_1,a_2}(t)dt. \end{cases}$$

Setting $R_{N,a_1,a_2}(t) := L\varsigma_{N,a_1,a_2}(t)$ in the above equation, we get

$$w(s) - \varsigma_{N,a_1,a_2}(s) = \int_a^b G(s,t)\left[R(t) - R_{N,a_1,a_2}(t)\right]dt. \qquad (8.7)$$

On the other hand, we set

$$f_{i,a_1,a_2}(t) := R(t) - R_{N,a_1,a_2}(t), \quad t \in [x_i, x_{i+1}], \quad i = 1,2,...,n \ , \qquad (8.8)$$

and consider an approximation of $f_{i,a_1,a_2}(t)$ by the second degree Newton's interpolation polynomial with nodes at $z_{i,1}$ and $z_{i,2}$:

$$f_{i,a_1,a_2}(t) = f_{i,a_1,a_2}[z_{i,1}] + f_{i,a_1,a_2}[z_{i,1}, z_{i,2}](t - z_{i,1}) + f_{i,a_1,a_2}[z_{i,1}, z_{i,2}, t](t - z_{i,1})(t - z_{i,2}) .$$

where $f_{i,a_1,a_2}[z_{i,1}]$, $f_{i,a_1,a_2}[z_{i,1}, z_{i,2}]$ and $f_{i,a_1,a_2}[z_{i,1}, z_{i,2}, t]$ respectively denote the 1st, 2nd and 3rd order divided differences. Also, because of the constraints $L\varsigma_{N,a_1,a_2}(z_{i,k}) = R(z_{i,k}),\ i = 1,2,...,n;\ k = 1,2$ (cf (7.9)), we conclude that $f_{i,a_1,a_2}[z_{i,1}] = 0 = f_{i,a_1,a_2}[z_{i,1}, z_{i,2}]$. Thus, (8.8) reduces to

$$f_{i,a_1,a_2}(t) = f_{i,a_1,a_2}[z_{i,1}, z_{i,2}, t](t - z_{i,1})(t - z_{i,2}) . \qquad (8.9)$$

Next, we set $W_{i,a_1,a_2}(s,t) := G(s,t) f_{i,a_1,a_2}[z_{i,1}, z_{i,2}, t],\ i = 1,2,...,n$. Then (8.7) can be expressed as (cf (8.8) and (8.9))

$$w(s) - \varsigma_{N,a_1,a_2}(s) = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} W_{i,a_1,a_2}(s,t)(t - z_{i,1})(t - z_{i,2})dt \ . \qquad (8.10)$$

The integral in (8.10) after using the second order Taylor approximation of $W_{i,a_1,a_2}(s,t)$ about $t = z_{i,1}$ can be written as

$$\int_{x_i}^{x_{i+1}} W_{i,a_1,a_2}(s,t)(t - z_{i,1})(t - z_{i,2})dt \tag{8.11}$$

$$= \int_{x_i}^{x_{i+1}} \left[W_{i,a_1,a_2}(s,z_{i,1}) + \frac{\partial W_{i,a_1,a_2}(s,z_{i,1})}{\partial t}(t - z_{i,1}) + \frac{\partial^2 W_{i,a_1,a_2}(s,\tau_i)}{2\partial t^2}(t - z_{i,1})^2 \right](t - z_{i,1})(t - z_{i,2})dt.$$

Since the polynomials $(t - z_{i,1})$ and $(t - z_{i,1})(t - z_{i,2})$ are mutually orthogonal over $[x_i, x_{i+1}]$ (cf (5.3)), the middle term in the above integral vanishes, i.e.,

$$\int_{x_i}^{x_{i+1}} \left[\frac{\partial W_{i,a_1,a_2}(s,z_{i,1})}{\partial t}(t - z_{i,1}) \right](t - z_{i,1})(t - z_{i,2})dt = 0. \tag{8.12}$$

However, the third term by use of Schwartz inequality and slight algebraic manipulations leads to

$$\int_{x_i}^{x_{i+1}} \frac{\partial^2 W_{i,a_1,a_2}(s,\tau_i)}{2\partial t^2}(t - z_{i,1})^3(t - z_{i,2})dt \le$$

$$\le \left\{ \int_{x_i}^{x_{i+1}} \left[\frac{\partial^2 W_{i,a_1,a_2}(s,\tau_i)}{2\partial t^2} \right]^2 dt \right\}^{0.5} \left\{ \int_{x_i}^{x_{i+1}} \left[(t - z_{i,1})^3(t - z_{i,2}) \right]^2 dt \right\}^{0.5} \tag{8.13}$$

$$\le \beta_2 h^4 \sqrt{x_{i+1} - x_i}.$$

This is possible because of the hypothesis of Theorem 8.1 related to *R*(*t*), i.e., we can fix a positive constant β_2 independent of *i*, *n*, *s*, a_1, a_2 and τ_i such that $\int_{x_i}^{x_{i+1}} \left[\frac{\partial^2 W_{i,a_1,a_2}(s,\tau_i)}{2\partial t^2} \right]^2 dt \le \beta_2$. In a similar manner, another positive constant β_1 exists such that $\left| W_{i,a_1,a_2}(s,z_{i,1}) \right| \le \beta_1$. Thus, the first term in the integral on the right side of (8.11) is estimated as

$$\int_{x_i}^{x_{i+1}} W_{i,a_1,a_2}(s,z_{i,1})(t - z_{i,1})(t - z_{i,2})dt \le \beta_1 \int_{x_i}^{x_{i+1}} (t - z_{i,1})(t - z_{i,2})dt. \tag{8.14}$$

Note that $\sum_{i=1}^{n} \sqrt{x_{i+1} - x_i} = n\sqrt{h}$. Then using (8.12)-(8.14) in (8.11) and inserting the resultant estimate in (8.10), we have

$$w(s) - \varsigma_{N,a_1,a_2}(s) \le \beta_1 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (t - z_{i,1})(t - z_{i,2})dt + \beta_2 (b-a) h^{3.5}.$$

The right hand side of the above inequality is independent of s. Therefore,

$$\left\| w - \varsigma_{N,a_1,a_2} \right\|_\infty = \max_{s\in[a,b]} \left| w(s) - \varsigma_{N,a_1,a_2}(s) \right| \le \beta_1 \sum_{i=1}^{n} \left| \int_{x_i}^{x_{i+1}} (t - z_{i,1})(t - z_{i,2})dt \right| + o(h^{3.5}) \qquad (8.11)$$

Finally, applying the limit $a_1 \to \frac{-1}{\sqrt{3}}$ on both sides of above inequality and then invoking Lemma 8.2, we note that each term in the $\sum$ notation on the right side of (8.11) will vanish. This leads to the desired conclusion, i.e.,

$$\lim_{a_1 \to \frac{-1}{\sqrt{3}}} \left\| w - \varsigma_{N,a_1,a_2} \right\|_\infty = O(h^{3.5}).$$

Corrolary 8. 1. If $a_1 = {-1}/{\sqrt{3}}$ in Theorem 8.1, in which case, the orthogonal pair (a_1,a_2) consists of the zeros of the 2nd degree Legendre polynomial, then

$$\left\| w - \varsigma_{N,a_1,a_2} \right\|_\infty = O(h^{3.5}).$$

8.4 A note on the order of approximation error

In Corollary 8.1, we have fixed an omission which appears in the monograph of Prenter [49]. In fact, the approximate error subject to the hypothesis of Theorem 8.1 for "*OCM based on Gaussian points*" is of order $h^{3.5}$ and not h^4 as concluded in [49, p.308, (13)].

9. Computational Aspects

This chapter explains the computational procedure for finding the approximate solution of two-point BVP

$$\begin{cases} w'' = P(t)w' + Q(t)w + R(t), & a < t < b, \\ w(a) = 0, & w(b) = 0. \end{cases} \tag{9.1}$$

by the orthogonal collocation method (*OCM*) based on the orthogonal pairs in [–1,1]. The procedure is implemented in the MATLAB 7.0 environment and is outlined in the following sections.

9.1 Input

The procedure is dependent on the following inputs:

a. *P, Q, R*: Parameters of BVP: (cf (9.1))

b. $[a,b]$: Solution interval (cf (9.1))

c. *N*: Number of subintervals of $[a,b]$

d. $a = x_1 < x_2 < \ldots < x_{N+1} = b$: Partitioning points of $[a,b]$

e. a_1: Position of fixed zero in [–1,1] related to 2nd degree orthogonal 0-interpolant over [–1,1].

9.2 Structure of subprograms

Several subprograms were designed for different objectives, i.e.,

i. Evaluation of $(z_{i,1}, z_{i,2})$, the pair of orthogonal zeros (collocation points) in each of the *N* subintervals $[x_i, x_{i+1}]$ based on the choice of input "a_1".

ii. Construction of four basic cubic Hermite polynomials: $\psi_U, \psi_L, \phi_L, \phi_R$ on the subintervals $[x_i, x_{i+1}]$ (cf (6.3)-6.6)). These are required to construct the 2*N* basis functions that involve the piecewise C-H polynomials with nodes at the partitioning points of $[a,b]$.

iii. Identifying the penta-diagonal coefficient matrix from the system of collocation equations.

iv. Solving the system of $2N$ linear equations (cf (7.9)) where right hand side of each equation is given as a value of R at the collocation points defined in (i) This program determines $2n$ unknowns c_i's required in the "linear combination of basis functions (cf (7.5)-(7.9)).

v. Construction of approximate solution.

vi. Evaluation of point-wise errors based on OCM with Legendre or *OZI* zeros.

vii. Comparison of point-wise, maximum and root-mean squared errors due to various OCM's, FDM and Shooting Method (see Sec. 9.3).

9.3 Subprograms of methods for comparative study

The following subprograms are written for the existing methods in order to make comparison of the results obtained by *OCM* based on the zeros of *OZI*:

(i) Standard *OCM* based on Gaussian points (*MG*),

(ii) Approximate solution by Finite Difference Method (*FDM*) using MATLAB built in program "bvp2c"

(iii) Approximate solution by Linear Shooting Method (*MS*) using MATLAB built in program "bvp4c"

9.4 Comparison of methods

Applying four approximating methods, namely, the OCM based on an orthogonal pair (a_1, a_2) in $[-1,1]$ (cf Section 9.2), Standard OCM, i.e., the OCM based on Gaussian knots (cf ((i), Section 9.3)), the FDM (cf ((ii), Section 9.3)) and the SM (cf (iii), Section 9.3) to a BVP, we have compared the resultant solutions of selective examples of the BVP. In this regard, we merely considered those values of the approximate solutions which were computed at the partitioning points while applying our proposed method, i.e., OCM based on an orthogonal pair.

9.5. Measure of approximation error

When an approximation is different from the exact value by a few percent, it makes sense to discuss the accuracy in terms of a relative and percentage errors. If the percentage error of two approximations is significantly different, it is reliable to use the percentage error to compare the accuracy of the two approximations. On the other hand, sometimes we want to compare the accuracy of two approximations where the difference between the resulting approximate values due to different methods and the exact value turns out a small fraction of a percent. In such a situation, it is useful to see *how much better* (or *worse)* one is than the other. For this, we shall use the measure of comparative accuracy for two different methods at the mesh points x_i as defined in (10.3) of the following chapter.

9.6 Computational environment and programs

We performed computation to obtain error of approximation and graphs in the MATLAB environment. The MATLAB functions constructed to achieve these objectives are attached in a separate folder.

10. Simulation Results

We have tested the proposed method on a few BVP's of the form

$$\begin{cases} w'' = P(t)w' + Q(t)w + R(t), & a < t < b, \\ w(a) = 0, & w(b) = 0. \end{cases} \tag{10.1}$$

with known solution. We compared the resulting solutions not only for different choices of $a_1 \in [-1,1]$ but also with those obtained by the *FDM*, the linear shooting method and the OCM with Gaussian knots.

We simulated the BVP's with different type of coefficients or forcing functions. Some examples of the form

$$-\frac{d}{dt}\left(p(t)\frac{dy}{dt}\right) + q(t)y = r(x). \tag{10.2}$$

Were also testd.

10.1 Optimal choice of orthogonal pair (a_1, a_2)

The approximation error due to OCM based on orthogonal pair (a_1, a_2) depends on the choice of nonzero point $a_1 \in [-1,1]$. It is one of the objectives of our project to observe an impact of various values of a_1 on the respective approximation error. *It is believed that OCM relevant to Legendre zeros, i.e., with* $a_1 = {-1}/{\sqrt{3}}$ *, is an optimal choice that provides the minimal approximation error.* Nevertheless, we found negation to this concept while dealing with specific choices of $a_1 \in [-1,1]$. While applying the simulation process on various examples, we determined nearly optimal location of the a_1 in the sense that it provides a best approximate solution as compared to other choices including $a_1 = {-1}/{\sqrt{3}}$.

10.2 A note on Simulation results

As stated in Section 9.5, we looked at the measure of comparative accuracy at the mesh points x_i. This measure denoted by $C_A(x_i)$ is defined as

$$C_A(x_i) := \left| [E(x_i) - MC(x_i)] \Big/ [E(x_i) - S_{a^*}(x_i)] \right| \tag{10.3}$$

where

E	:=	exact value,
S_{a^*}	:=	approximate value due to *OCM* based on orthognal pair (a_1, a_2) with an optimal value $a_1 = a^*$,
MC	:=	approximate value based on a method to be compared with S_{a^*}.

We shall use the following abbreviations while explaining the simulation results for test examples:

MZ	:=	*OCM* based on an *orthogonal pair of points*.
MF	:=	*FDM* using MATLAB function bvp2c.
MG	:=	*OCM* with *Gaussian knots*.
MS	:=	*Shooting method* using MATLAB function bvp4c.
RMS	:=	*Root mean squared error* corresponding to given mesh for relevant approximation method
MPE	:=	*Maximum point-wise error* corresponding to given mesh

Remark 10.1. Note that $C_A(x_i)$ determines the level of superiority of suggested method when compared with another approximating method under consideration, i.e., $C_A(x_i) > 1$ will imply that S_{a^*} is a better approximation than relevant *MC* at x_i, and otherwise if $C_A(x_i) < 1$. The magnitude of $C_A(x_i)$, when greater than 1, tells us how much S_{a^*} is better than the *MC* under consideration.

While applying various computational methods on a selective BVP, we discussed

- some of the characteristics of the BVP and its exact solution.
- $C_A(x_i)$ for *MZ* (different choices of $a_1 \in [-1, 1]$) versus S_{a^*}.
- $C_A(x_i)$ for *MF* versus S_{a^*}.
- $C_A(x_i)$ for *MG* versus S_{a^*}.
- $C_A(x_i)$ for *MS* versus S_{a^*}.

Description of Tables: We have discussed $C_A(x_i)$ for *MZ* with different choices of $a_1 \in [-1,1]$ versus S_{a^*} in case of Examples 1-2. The resultaning errors have been provided in different Tables corresponding to 6, 16, 26 uniform mesh points. These tables indicate the gradual reduction of error when a_1 gets closer to certain optimal value a^*. Here, we have also included *RMS* and *MPE* corresponding to each a_1 under consideration. For rest of the example, we computed $C_A(x_i)$, *RMS* and *MPE* only corresponding to 26 mesh points.

Description of Graphs: Based on 25 mesh points we have provided comparison of point-wise errors due to

- *MF* and S_{a^*} in a graph only for Examples 1-2.
- *MG* and S_{a^*} in a graph for all test examples.
- *MS* and S_{a^*} in a graph for all test examples

10.3 Simulation results for Example 1

(BVP) $y'' - y' - 6y = \sin x; \quad y(0) = y(2\pi) = 0$

(Solution) $y(x) = \dfrac{1}{50(e^{6\pi} - e^{-4\pi})}\left[(1-e^{6\pi})e^{-2x} - (1-e^{-4\pi})e^{3x}\right] + \dfrac{1}{50}(\cos x - 7\sin x)$

Characteristics of the BVP: It is a simple BVP with constant coefficients with the presence of first derivative term, i.e., this BVP is not of the type (10.2). Its exact solution involves three transcendental functions and the number π.

Tables 10.3.1-10.3.3 provide Comparative Accuracy for *different choices* of $a_1 \in [-1,1]$ versus an *optimal choice* of $a^* = -0.57843961$ for Example 1.

Table 10.3.1: Magnitude of $C_A(x_i)$ w.r.t. a^* [5 Subintervals]

$x_i = 2\pi *$	$a_1 = -1$	$a_1 = -0.6$	$a_1 = -0.5$	$a_1 = -0.4$	$a_1 = 0.5$	$a_1 = 1$		$a^* = -0.57843961$
0.2	2.0876	0.8037	1.7621	2.8372	5.1215	1.8648		**1.0000**
0.4	1.3825	0.8396	1.6565	2.6959	4.9230	5.1699		**1.0000**
0.6	2.3344	0.8334	1.5907	2.2887	6.4016	1.2014		**1.0000**
0.8	10.5274	1.6946	1.4148	3.6155	5.5536	31.1304		**1.0000**
MPE	7.966 e-004	3.067 e-004	6.724 e-004	0.0011	0.0020	0.0014		**3.816 e-004**
RMS Error	4.5175 e-004	1.6462 e-004	3.3941 e-004	5.3810 e-004	0.0011	7.4983 e-004		**1.9903 e-004**

Table 10.3.2: Magnitude of $C_A(x_i)$ w.r.t. a^* [10 Subintervals]

$x_i = 2\pi *$	$a_1 = -1$	$a_1 = -0.6$	$a_1 = -0.5$	$a_1 = -0.4$	$a_1 = 0.5$	$a_1 = 1$		$a^* = -0.57843961$
0.1	11.3868	0.1124	4.5050	9.5426	18.5515	11.1508		**1.0000**
0.2	8.7134	0.3150	3.6982	7.5619	14.8537	7.7912		**1.0000**
0.3	8.2762	0.3459	3.5808	7.2937	14.3018	8.1530		**1.0000**
0.4	9.1439	0.2729	3.8880	8.1071	15.6301	11.6366		**1.0000**
0.5	38.4692	4.2007	12.4753	34.5381	54.8327	141.7843		**1.0000**
0.6	5.4783	0.5775	2.6195	4.8073	10.2953	0.5388		**1.0000**
0.7	6.3155	0.5130	2.8892	5.5278	11.7131	4.6390		**1.0000**
0.8	3.9701	0.6997	2.1830	3.9897	9.2609	11.7307		**1.0000**
0.9	13.4234	0.0324	4.6601	9.3751	21.7258	2.7236		**1.0000**
MPE	3.251 e-004	1.046 e-005	1.129 e-004	2.27 e-004	5.262 e-004	1.846 e-004		**2.422 e-005**
RMS Error	1.5081 e-004	5.8299 e-006	5.9788 e-005	1.2102 e-004	2.539 e-004	1.1925 e-004		**1.5880 e-005**

Table 10.3.3: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

$x_i = 2\pi *$	$a_1 = -1$	$a_1 = -0.6$	$a_1 = -0.5$	$a_1 = -0.4$	$a_1 = 0.5$	$a_1 = 1$		$a^* = -0.57843961$
0.04	169.3383	11.5250	50.6289	122.3299	253.1199	177.6554		**1.0000**
0.08	112.1855	7.2910	33.8174	81.1282	168.8035	113.5900		**1.0000**
0.12	85.7250	5.3316	26.0391	62.0711	129.7547	84.2481		**1.0000**
0.16	72.3622	4.3432	22.1172	52.4687	110.0211	69.7311		**1.0000**
0.20	65.5889	3.8434	20.1366	47.6272	100.0026	62.6913		**1.0000**
0.24	62.5459	3.6206	19.2564	45.4857	95.4809	59.9206		**1.0000**
0.28	61.9104	3.5769	19.0886	45.0940	94.5026	59.9678		**1.0000**
0.32	63.1912	3.6762	19.4906	46.1052	96.3384	62.3610		**1.0000**
0.36	66.6129	3.9361	20.5339	48.6992	101.3092	67.5332		**1.0000**
0.40	73.7334	4.4746	22.6908	54.0479	111.6833	77.7185		**1.0000**
0.44	91.5355	5.8186	28.0708	67.3768	137.6466	102.6495		**1.0000**
0.48	204.1714	14.3175	62.0835	151.6134	301.9793	259.0647		**1.0000**
0.52	39.4782	4.0651	11.4799	30.5617	53.5245	78.6181		**1.0000**
0.56	16.6835	0.1728	5.4810	11.4465	28.4096	0.4805		**1.0000**
0.60	30.6462	1.2271	9.7020	21.9064	48.7699	19.2373		**1.0000**
0.64	37.2535	1.7271	11.7058	26.8805	58.3890	28.9701		**1.0000**
0.68	41.1400	2.0233	12.8973	29.8530	64.0168	35.3798		**1.0000**
0.72	43.4289	2.2028	13.6285	31.7103	67.2606	40.5729		**1.0000**
0.76	43.9743	2.2612	13.8942	32.4804	67.8151	45.8955		**1.0000**
0.80	40.2511	2.0362	13.0958	30.8513	61.6053	54.2011		**1.0000**
0.84	6.6538	1.0037	1.8234	5.9500	13.7334	101.4763		**1.0000**
0.88	88.4089	5.1909	24.8727	57.1772	138.4742	17.7331		**1.0000**
0.92	73.9363	4.2527	21.3968	49.5194	115.2253	34.0552		**1.0000**
0.96	71.5332	4.0958	20.8178	48.2636	111.3731	38.7619		**1.0000**
MPE	5.277 e-005	3.022 e-006	1.536 e-004	3.560 e-005	8.216 e-005	3.059 e-005		**7.377 e-007**
RMS Error	2.3499 e-005	1.3680 e-006	7.0924 e-006	1.6667 e-005	3.609 e-005	2.0088 e-005		**3.6559 e-007**

Table 10.3.4 provides Comparative Accuracy when OCM based on an optimal choice $a^* = -0.57843961$ is compared separately with the methods "*FDM*, *Shooting Method* and *OCM based on Gaussian Knots*" for Example 1

Table 10.3.4: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

$x_i = 2\pi *$	FDM	Shooting Method	OCM Gaussian Knots		OCM with $a^* = -0.57843961$
0.04	2096.4	6.5934	1.6457		**1.0000**
0.08	1332.0	17.6068	1.4273		**1.0000**
0.12	936.6	10.9403	1.3263		**1.0000**
0.16	697.8	9.9438	1.2753		**1.0000**
0.20	535.4	2.1540	1.2495		**1.0000**
0.24	411.7	3.0665	1.2380		**1.0000**
0.28	304.8	4.1155	1.2358		**1.0000**
0.32	197.8	3.2530	1.2409		**1.0000**
0.36	070.0	10.6261	1.2544		**1.0000**
0.40	121.7	8.7176	1.2822		**1.0000**
0.44	534.9	3.1688	1.3516		**1.0000**
0.48	3001.1	26.0798	1.7906		**1.0000**
0.52	275.7	7.2476	0.8411		**1.0000**
0.56	43.2	0.5904	1.0600		**1.0000**
0.60	728.4	3.0633	1.1145		**1.0000**
0.64	572.9	3.6331	1.1403		**1.0000**
0.68	473.6	10.1071	1.1556		**1.0000**
0.72	4006	1.3655	1.1649		**1.0000**
0.76	343.1	0.6848	1.1680		**1.0000**
0.80	298.0	0.0986	1.1567		**1.0000**
0.84	290.9	3.3850	1.0020		**1.0000**
0.88	188.2	3.1191	1.3174		**1.0000**
0.92	170.1	1.5642	1.2697		**1.0000**
0.96	148.9	0.3433	1.2617		**1.0000**
MPE	3.051 e-004	4.589 e-006	9.308 e-007		**7.377 e-007**
RMS Error	1.7481 e-004	2.0272 e-006	4.5093 e-007		**3.6559 e-007**

Table 10.3.5 provides point-wise errors over the 26 uniform mesh points for the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots and OCM based on an optimal choice of a**" when applied to Example 1

Table 10.3.5: Point-wise errors [25 Subintervals]

$x_i =$ 2π *	**FDM** 1.0e-003 *	**Shooting Method** 1.0e-005 *	**OCM Gaussian Knots** 1.0e-006 *		**OCM with** $a^* = -0.57843961$ 1.0e-006 *
0.00	0	0	0		0
0.04	0.2003	0.0630	0.1572		0.0956
0.08	0.2838	0.3752	0.3041		0.2131
0.12	0.3051	0.3564	0.4321		0.3258
0.16	0.2918	0.4158	0.5333		0.4182
0.20	0.2577	0.1037	0.6014		0.4813
0.24	0.2102	0.1565	0.6320		0.5105
0.28	0.1537	0.2076	0.6232		0.5043
0.32	0.0917	0.1509	0.5755		0.4638
0.36	0.0275	0.4167	0.4919		0.3921
0.40	0.0358	0.2567	0.3775		0.2944
0.44	0.0949	0.0562	0.2398		0.1774
0.48	0.1465	0.1273	0.0874		0.0488
0.52	0.1880	0.0599	0.0695		0.0826
0.56	0.2168	0.0123	0.2203		0.2078
0.60	0.2314	0.0973	0.3540		0.3176
0.64	0.2308	0.1464	0.4594		0.4028
0.68	0.2150	0.4589	0.5247		0.4540
0.72	0.1846	0.0629	0.5368		0.4608
0.76	0.1411	0.0282	0.4803		0.4112
0.80	0.0865	0.0029	0.3357		0.2902
0.84	0.0239	0.0278	0.0822		0.0821
0.88	0.0411	0.0682	0.2879		0.2186
0.92	0.0962	0.0885	0.7181		0.5656
0.96	0.1099	0.0253	0.9308		0.7377
1.00	0.000	0.0000	0.0000		0.0000
	0				
MPE	3.051 e-004	4.589 e-006	9.308 e-007		**7.377 e-007**
RMS Error	1.7481 e-004	2.0272 e-006	4.5093 e-007		**3.6559 e-007**

Figure 10.3.1 **(Example 1)**

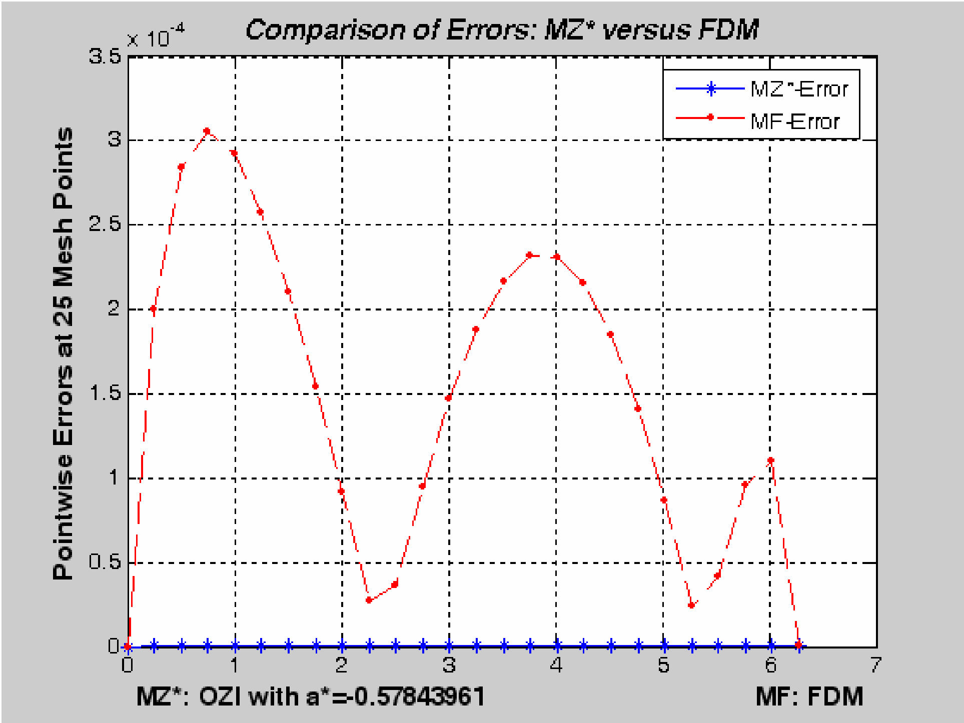


Figure 10.3.2 **(Example 1)**

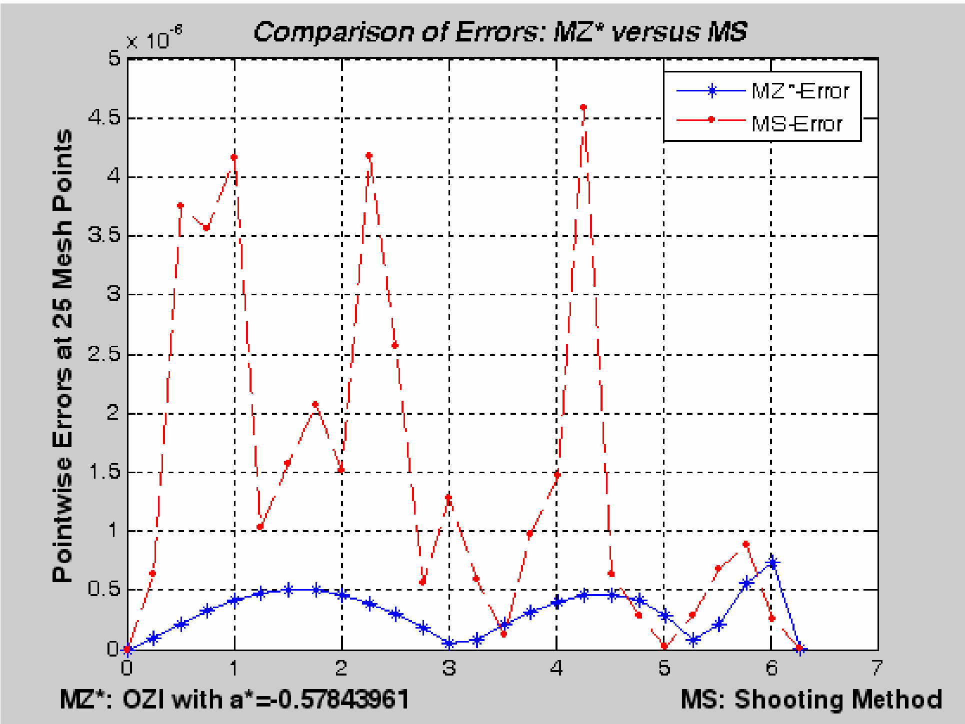
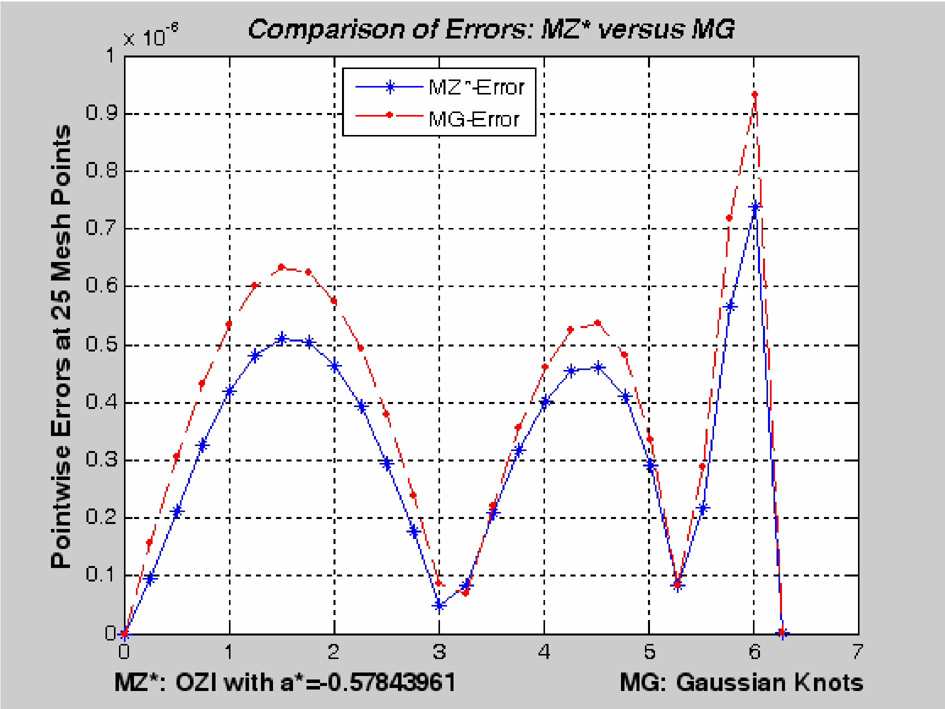


Figure 10.3.3 (Example 1)



10.4 Simulation results for Example 2

(BVP) $y'' - 400y = 400\cos^2(\pi x) + 2\pi^2\cos(2\pi x);\quad y(0) = y(1) = 0$

(Solution) $y(x) = \dfrac{1}{1+e^{-20}}\left[e^{20(x-1)} + e^{-20x}\right] + \cos^2(\pi x)$

Characteristics of the BVP: It is also a simple BVP with constant coefficients. However, its associated homogeneous differential equation has solutions of the form $y = Ce^{\pm 20x}$ which grow or decay at a rapid exponential rate. In addition, the BVP is of the type (10.2) with $p(t) = 1$. Its exact solution involves two transcendental functions and the number π.

Tables 10.4.1-10.4.3 provide Comparative Accuracy for *different choices* of $\boldsymbol{a}_1 \in [-1,1]$ versus an *optimal choice* of $\boldsymbol{a}^* = -\mathbf{0.57843961}$ in case of Example 2.

Table 10.4.1: Magnitude of $C_A(x_i)$ w.r.t. a^* [5 Subintervals]

x_i	$a_1 = -1$	$a_1 = -0.6$	$a_1 = -0.5$	$a_1 = -0.4$	$a_1 = 0.5$	$a_1 = 1$		a^* $= -0.57843961$
0.2	2.4384	0.9621	1.2356	1.7611	2.2582	4.2689		**1.0000**
0.4	0.1919	0.8808	1.5004	2.3168	1.7856	0.1788		**1.0000**
0.6	0.1793	0.8320	1.7906	3.1800	1.5046	0.1925		**1.0000**
0.8	4.3343	0.6555	2.2928	3.9255	1.2546	2.4757		**1.0000**
MPE	0.0428	0.0096	0.0226	0.0388	0.0226	0.0245		**0.0100**
RMS	0.0201	0.0048	0.0106	0.0176	0.0106	0.0201		**0.0058**

Table 10.4.2: Magnitude of $C_A(x_i)$ w.r.t. a^* [10 Subintervals]

x_i	$a_1 = -1$	$a_1 = -0.6$	$a_1 = -0.5$	$a_1 = -0.4$	$a_1 = 0.5$	$a_1 = 1$		a^* $= -0.57843961$
0.1	3.6078	0.5123	3.0596	6.3558	5.3325	16.0529		1.0000
0.2	3.7399	0.4946	3.1550	6.6750	5.4874	14.8630		1.0000
0.3	4.5224	0.5042	3.0681	6.3519	4.4318	9.8803		1.0000
0.4	4.6600	0.5908	2.6342	5.0465	2.7515	4.7004		1.0000
0.5	4.2331	0.6299	2.4731	4.6321	2.4731	4.2331		1.0000
0.6	4.7064	0.5669	2.7550	5.3918	2.6376	4.6660		1.0000
0.7	10.0651	0.1306	4.5147	9.7017	3.1255	4.6070		1.0000
0.8	15.3659	0.1892	5.6731	12.2104	3.2617	3.8665		1.0000
0.9	16.5984	0.1793	5.5137	11.5382	3.1636	3.7304		1.0000
Max Error	0.0197	6.285 e-004	0.0065	0.0137	0.0065	0.0197		0.0012
RMS Error	0.0063	2.0882 e-004	0.0024	0.0049	0.0024	0.0063		5.3560 e-004

Table 10.4.3: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

x_i	$a_1 = -1$	$a_1 = -0.6$	$a_1 = -0.5$	$a_1 = -0.4$	$a_1 = 0.5$	$a_1 = 1$		a^* $= -0.57843961$
0.04	71.2262	4.6189	23.5936	57.1655	31.0996	108.8841		1.0000
0.08	71.8544	4.6812	23.8659	57.9050	31.4990	109.8777		1.0000
0.12	72.8062	4.7693	24.2419	58.9043	32.0317	111.3323		1.0000
0.16	73.9490	4.8709	24.6714	60.0334	32.5930	112.8413		1.0000
0.20	74.4653	4.9164	24.8672	60.5591	32.6964	112.6097		1.0000
0.24	71.9867	4.7071	24.0139	58.4068	31.0563	106.0065		1.0000
0.28	63.5188	3.9966	21.1030	51.0201	26.2269	88.0109		1.0000
0.32	51.1089	2.9637	16.8742	40.2949	19.5989	63.9328		1.0000
0.36	41.0592	2.1358	13.4910	31.7325	14.5450	45.8773		1.0000
0.40	35.3423	1.6718	11.6021	26.9707	11.8899	36.5562		1.0000
0.44	32.5436	1.4501	10.7048	24.7244	10.7387	32.6127		1.0000
0.48	31.3544	1.3595	10.3421	23.8275	10.3350	31.2903		1.0000
0.52	31.2866	1.3567	10.3338	23.8132	10.3409	31.3507		1.0000
0.56	32.6263	1.4591	10.7433	24.8217	10.7093	32.5573		1.0000
0.60	36.7016	1.7604	11.9372	27.7307	11.6482	35.4829		1.0000
0.64	46.5643	2.4775	14.7628	34.5717	13.6930	41.6740		1.0000
0.68	66.4844	3.9090	20.3811	48.1119	17.5477	53.1487		1.0000
0.72	94.8822	5.9278	28.2745	67.0512	22.7506	68.4779		1.0000
0.76	117.7545	7.5286	34.4981	81.8859	26.6752	79.9646		1.0000
0.80	126.6878	8.1267	36.7840	87.2264	27.9761	83.7747		1.0000
0.84	127.1637	8.1256	36.7299	86.9607	27.8029	83.3350		1.0000
0.88	125.2243	7.9574	36.0286	85.1601	27.2668	81.8909		1.0000
0.92	123.3030	7.7933	35.3477	83.4204	26.7819	80.6339		1.0000
0.96	121.9634	7.6719	34.8354	82.0909	26.4277	79.7820		1.0000
MPE	0.0029	1.839 e-004	8.349 e-004	0.0020	8.349 e-004	0.0029		2.685 e-005
RMS Error	0.0010	6.6661 e-005	3.1473 e-004	7.5032 e-004	3.1473 e-004	0.0010		1.0727 e-005

Table 10.4.4 provides Comparative Accuracy when OCM based on an optimal choice $a^* = -0.57843961$ is compared separately with the methods "*FDM*, *Shooting Method* and *OCM based on Gaussian Knots*" for Example 2

Table 10.4.4: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

x_i	FDM	Shooting Method	OCM Gaussian Knots		OCM with $a^* = -0.57843961$
0.04	332.4239	0.4480	1.2906		**1.0000**
0.08	339.6638	0.3671	1.2939		**1.0000**
0.12	348.5407	0.3276	1.2985		**1.0000**
0.16	358.1379	0.8264	1.3038		**1.0000**
0.20	363.5037	2.2990	1.3062		**1.0000**
0.24	350.7480	3.8487	1.2953		**1.0000**
0.28	302.8619	6.3972	1.2584		**1.0000**
0.32	232.6606	11.7362	1.2048		**1.0000**
0.36	176.9006	15.9683	1.1618		**1.0000**
0.40	146.4062	2.9590	1.1378		**1.0000**
0.44	132.4947	5.3939	1.1263		**1.0000**
0.48	127.2494	2.5161	1.1216		**1.0000**
0.52	127.2341	2.5158	1.1215		**1.0000**
0.56	132.5503	5.3962	1.1268		**1.0000**
0.60	146.9888	2.9708	1.1423		**1.0000**
0.64	179.5495	16.2074	1.1792		**1.0000**
0.68	241.9463	12.2046	1.2529		**1.0000**
0.72	326.5071	6.8966	1.3567		**1.0000**
0.76	389.6192	4.2752	1.4389		**1.0000**
0.80	408.9478	2.5864	1.4695		**1.0000**
0.84	403.5946	0.9312	1.4693		**1.0000**
0.88	392.0315	0.3684	1.4605		**1.0000**
0.92	381.1654	0.4120	1.4520		**1.0000**
0.96	372.3550	0.5019	1.4456		**1.0000**
MPE	0.0089	2.693 e-005	3.465 e-005		**2.685 e-005**
RMS Error	0.0038	1.2426 e-005	1.4626 e-005		**1.0727 e-005**

Table 10.4.5 provides point-wise errors over the 26 uniform mesh points for the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots*" when applied to Example 2

Table 10.4.5: Point-wise errors [25 Subintervals]

x_i =	FDM	Shooting Method 1.0e-004 *	OCM Gaussian Knots 1.0e-004 *		OCM with $a^* = -0.57843961$ 1.0e-004 *
0.00	0	0	0		0
0.04	0.0089	0.1203	0.3465		0.2685
0.08	0.0081	0.0870	0.3067		0.2371
0.12	0.0054	0.0511	0.2027		0.1561
0.16	0.0033	0.0754	0.1189		0.0912
0.20	0.0019	0.1171	0.0665		0.0509
0.24	0.0010	0.1133	0.0381		0.0294
0.28	0.0006	0.1265	0.0249		0.0198
0.32	0.0004	0.1961	0.0201		0.0167
0.36	0.0003	0.2693	0.0196		0.0169
0.40	0.0003	0.0540	0.0208		0.0182
0.44	0.0003	0.1062	0.0222		0.0197
0.48	0.0003	0.0517	0.0230		0.0206
0.52	0.0003	0.0517	0.0230		0.0206
0.56	0.0003	0.1062	0.0222		0.0197
0.60	0.0003	0.0540	0.0208		0.0182
0.64	0.0003	0.2693	0.0196		0.0166
0.68	0.0004	0.1961	0.0201		0.0161
0.72	0.0006	0.1265	0.0249		0.0183
0.76	0.0010	0.1133	0.0381		0.0265
0.80	0.0019	0.1171	0.0665		0.0453
0.84	0.0033	0.0754	0.1189		0.0809
0.88	0.0054	0.0511	0.2027		0.1388
0.92	0.0081	0.0870	0.3067		0.2113
0.96	0.0089	0.1203	0.3465		0.2397
1.00	0.0000	0.0000	0.0000		0.0000
MPE	0.0089	2.693 e-005	3.465 e-005		**2.685 e-005**
RMS Error	0.0038	1.2426 e-005	1.4626 e-005		**1.0727 e-005**

Figure 10.4.1 **(Example 2)**

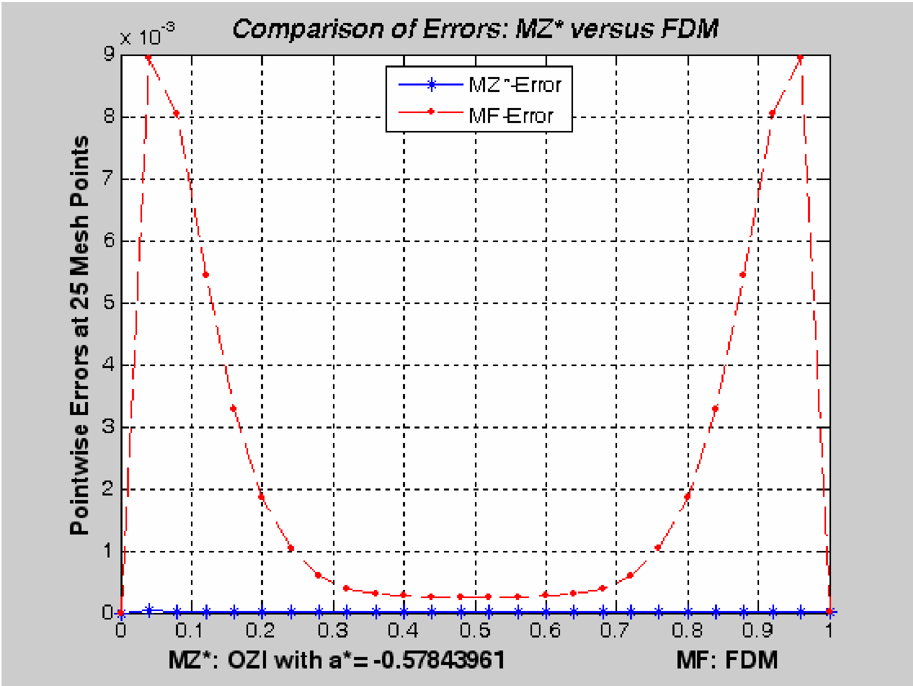


Figure 10.4.2 **(Example 2)**

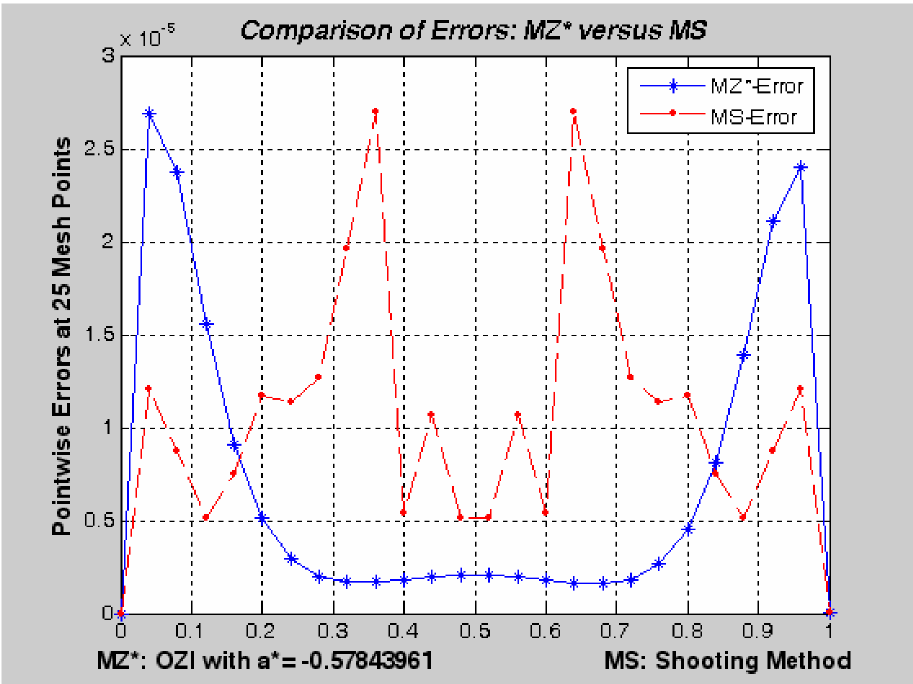
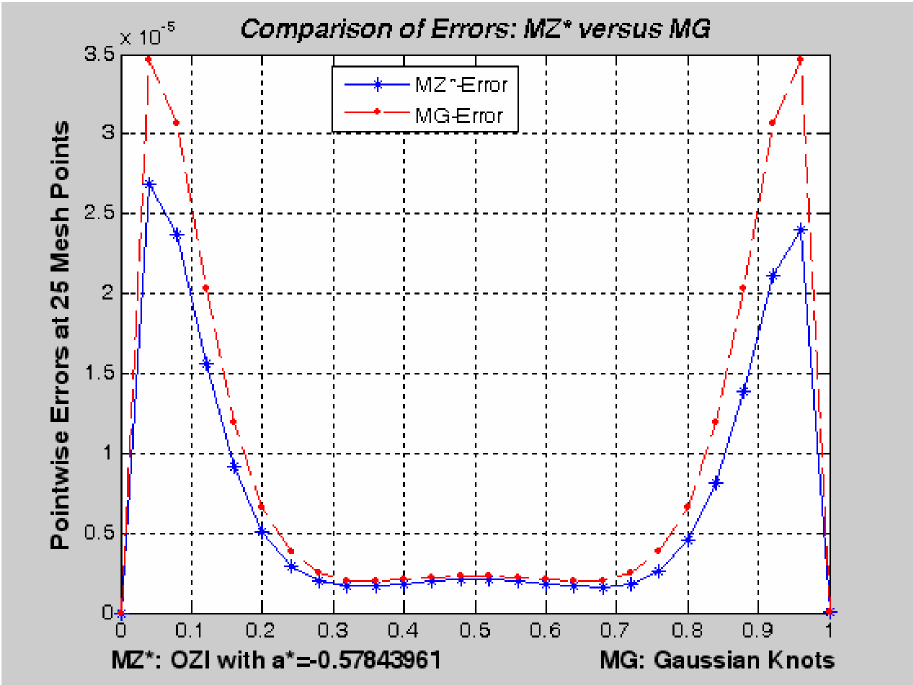


Figure 10.4.3 **(Example 2)**



10.5 Simulation results for Example 3

(BVP) $\left(e^{x} y'\right)' = -e^{-x}; \quad y(0) = y(b) = 0$

(Solution) $y(x) = \frac{1}{2}(1-e^{-x})(e^{-x} - e^{-b})$

Characteristics of the BVP: This BVP is of the type (10.2) with $p(t) = e^{x}$ and $q(t) = 0$. Here, the underlying interval of the BVP is [0,*b*]. The parameter *b* appears in the particular solution. The term e^{-b} decays with the increase in the length of the interval, i.e., the solution $y(x) \approx \frac{1}{2}(1-e^{-x})e^{-x}$ if *b* is very large. Also, the solution decays rapidly with the increase in *b*. The exact solution involves only one transcendental function. We have selected *b* = 1 for this problem.

Table 10.5.1 provides Comparative Accuracy when OCM based on an optimal choice $a^* = -0.57843961$ is compared separately with the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots and OCM based on* $a_1 = -0.57843961$ (*a best choice for Examples* 1-2)" for Example 3

Table 10.5.1: Magnitude of $C_A(x_i)$ w.r.t a^* [25 Subintervals]

x_i =	FDM 1.0e+005 *	Shooting Method 1.0e+005 *	OCM Gaussian Knots	OCM with a_1= –0.57843961	OCM with a^*= -0.57737961
0.04	0.9044	1.9502	18.8987	625.9051	**1.0000**
0.08	0.9342	1.1295	19.2091	635.5007	**1.0000**
0.12	0.9655	0.0582	19.5411	645.7258	**1.0000**
0.16	1.0025	0.3823	19.9755	659.2437	**1.0000**
0.20	1.0395	0.4790	20.3989	672.2966	**1.0000**
0.24	1.0831	0.0412	20.9370	689.0245	**1.0000**
0.28	1.1275	0.1496	21.4742	705.6183	**1.0000**
0.32	1.1739	0.3192	22.0349	722.8931	**1.0000**
0.36	1.2495	0.0254	23.1201	757.2599	**1.0000**
0.40	1.2791	0.0692	23.3372	763.0376	**1.0000**
0.44	1.2976	0.0763	23.3574	762.0490	**1.0000**
0.48	1.3995	0.0519	24.8752	809.2313	**1.0000**
0.52	1.4585	0.0309	25.6389	830.5533	**1.0000**
0.56	1.0070	0.1767	17.5052	564.8130	**1.0000**
0.60	0.9110	0.0416	15.6220	503.3706	**1.0000**
0.64	0.9586	0.0078	16.1871	521.8822	**1.0000**
0.68	1.0039	0.1566	16.7430	538.6411	**1.0000**
0.72	1.2950	0.0925	21.2926	684.8327	**1.0000**
0.76	1.3662	0.0349	22.1257	712.2108	**1.0000**
0.80	1.1313	0.1913	18.0764	581.5295	**1.0000**
0.84	1.1298	0.1486	17.7321	572.6670	**1.0000**
0.88	1.1342	0.0410	17.5058	567.0643	**1.0000**
0.92	1.0499	0.3227	15.9646	517.8384	**1.0000**
0.96	1.0806	0.5894	16.1969	525.8462	**1.0000**
MPE	0.6950 *1.0e-005	0.3207 *1.0e-005	0.1833 *1.0e-008	0.4146 *1.0e-007	**0.6674 *1.0e-010**
RMS Error	4.9307 *1.0e-006	1.1810 *1.0e-006	8.8998 *1.0e-010	2.9021 *1.0e-008	**4.3474 *1.0e-011**

Table 10.5.2 provides point-wise errors over the 26 uniform mesh points for the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots, OCM based on* a_1 **= –0.57843961** (*a best choice for Examples* 1-2) *and OCM based on an optimal choice of* a*" when applied to Example 3

Table 10.5.2: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

x_i =	FDM 1.0e-005 *	Shooting Method 1.0e-005 *	OCM Gaussian Knots 1.0e-008 *	OCM with a_1= −0.57843961 1.0e-007 *	OCM with a^* = −0.578210009 1.0e-010 *
0.00	0	0	0	0	0
0.04	0.1423	0.3068	0.0297	0.0985	0.1573
0.08	0.2652	0.3207	0.0545	0.1804	0.2839
0.12	0.3702	0.0223	0.0749	0.2476	0.3834
0.16	0.4583	0.1748	0.0913	0.3014	0.4572
0.20	0.5309	0.2446	0.1042	0.3433	0.5107
0.24	0.5889	0.0224	0.1138	0.3746	0.5437
0.28	0.6333	0.0841	0.1206	0.3964	0.5617
0.32	0.6652	0.1809	0.1249	0.4097	0.5667
0.36	0.6855	0.0140	0.1268	0.4155	0.5486
0.40	0.6950	0.0376	0.1268	0.4146	0.5434
0.44	0.6946	0.0408	0.1250	0.4079	0.5353
0.48	0.6850	0.0254	0.1218	0.3961	0.4895
0.52	0.6669	0.0141	0.1172	0.3798	0.4572
0.56	0.6410	0.1125	0.1114	0.3595	0.6366
0.60	0.6080	0.0278	0.1043	0.3359	0.6674
0.64	0.5684	0.0046	0.0960	0.3095	0.5930
0.68	0.5228	0.0816	0.0872	0.2805	0.5208
0.72	0.4718	0.0337	0.0776	0.2495	0.3643
0.76	0.4158	0.0106	0.0673	0.2168	0.3044
0.80	0.3553	0.0601	0.0568	0.1826	0.3141
0.84	0.2908	0.0383	0.0456	0.1474	0.2574
0.88	0.2226	0.0080	0.0344	0.1113	0.1963
0.92	0.1512	0.0465	0.0230	0.0746	0.1440
0.96	0.0769	0.0419	0.0115	0.0374	0.0712
1.00	0.0000	0.0000	0.0000	0.0000	0.0000
MPE	0.6950 *1.0e-005	0.3207 *1.0e-005	0.1833 *1.0e-008	0.4146 *1.0e-007	**0.6674 *1.0e-010**
RMS Error	4.9307 *1.0e-006	1.1810 *1.0e-006	8.8998 *1.0e-010	2.9021 *1.0e-008	**4.3474 *1.0e-011**

Figure 10.5.1 (Example 3)

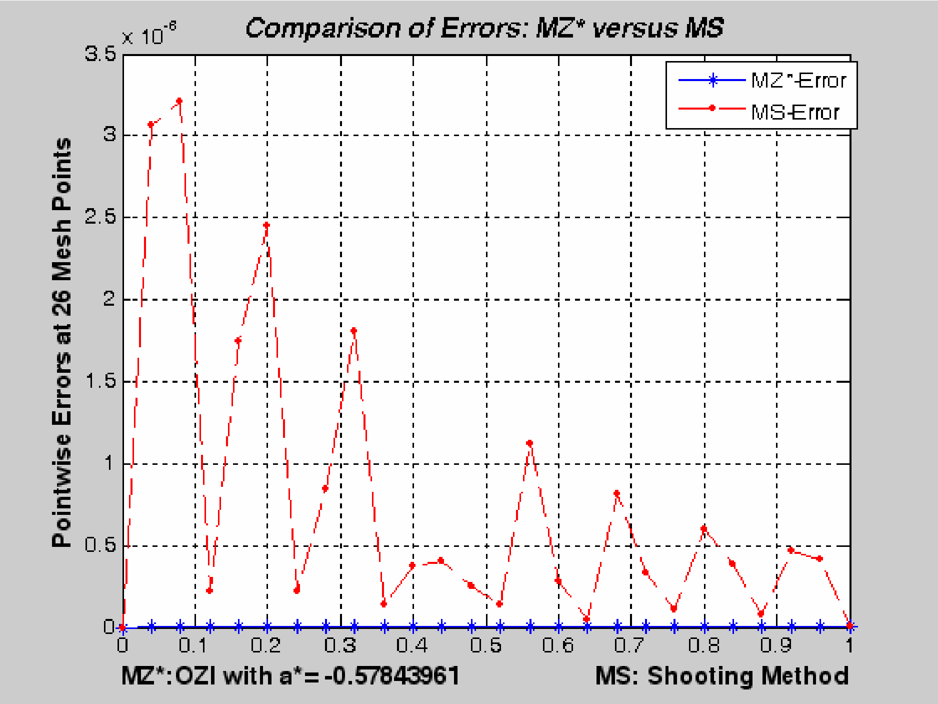


Figure 10.5.2 (Example 3)

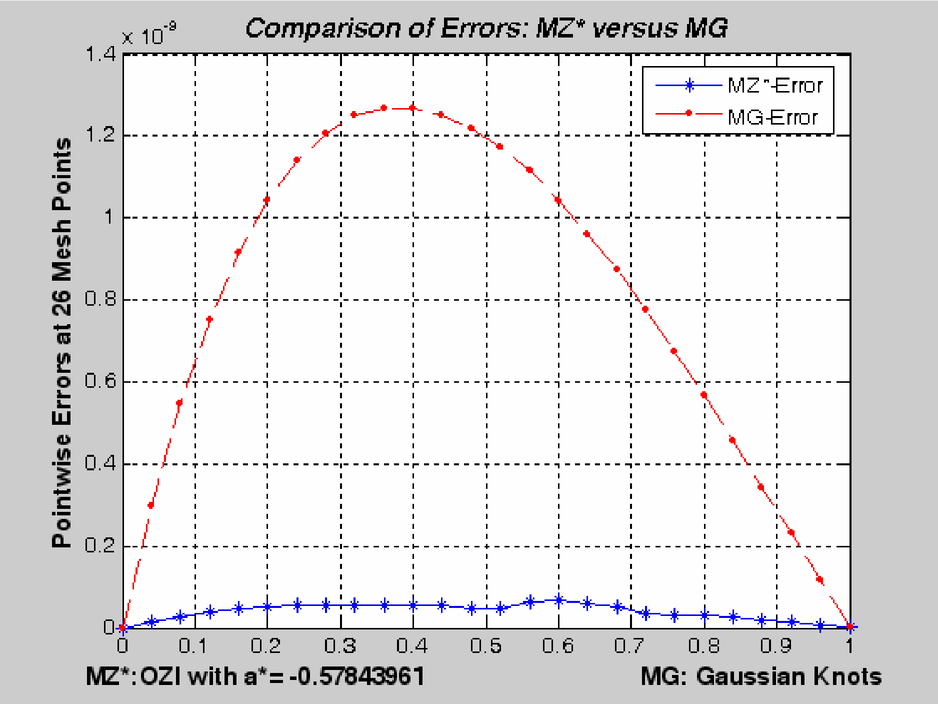
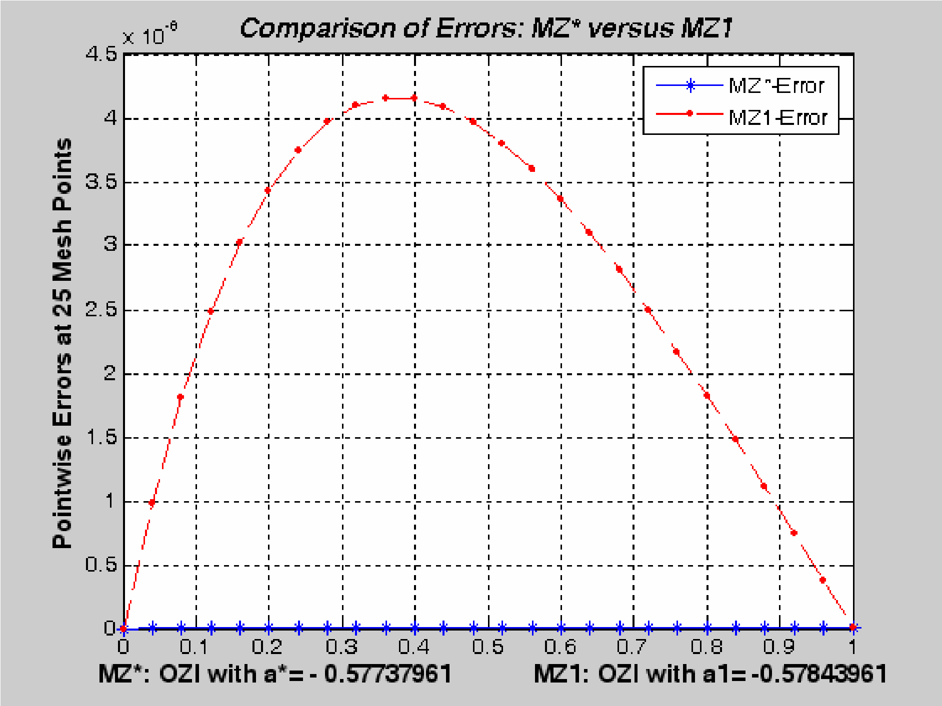


Figure 10.5.3 **(Example 3)**



10.6 Simulation results for Example 4

(BVP) $\left((x^2+1)y'\right)' - 2y = 1;\ y(0) = y(1) = 0$

(Solution) $y(x) = \dfrac{-\pi x}{8} + \dfrac{x}{2}\tan^{-1} x$

Characteristics of the BVP: This BVP is also of the type (10.2) with $p(t) = x^2 + 1$ and $q(t) = -2$. However, The exact solution involves a transcendental function as well as a linear polynomial.

Table 10.6.1 provides Comparative Accuracy when OCM based on an optimal choice $a^* = -0.577399$ is compared separately with the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots, OCM based on* $a_1 = -0.57843961$ (*a best choice for Examples* 1-2) and $a_3 = -0.57737961$ (a best choice for Example 3) for Example 4.

Table 10.6.1: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

x_i =	**FDM** 1.0e+007 *	**Shooting Method** 1.0e+004 *	**OCM Gaussian Knots**	**OCM *OZI-Knots*** $a_1 = -0.57843961$	**OCM *OZI-Knots*** $a_3 = -0.57737961$	**OCM with** $a^* = -0.577399$
0.04	0.2247	1.5907	1.2188	3.6736	1.0866	**1.0000**
0.08	0.2644	2.4118	1.2277	3.8632	1.0901	**1.0000**
0.12	0.3164	2.4566	1.2396	4.1199	1.0947	**1.0000**
0.16	0.3878	1.7649	1.2561	4.4744	1.1011	**1.0000**
0.20	0.4917	0.5142	1.2799	4.9851	1.1103	**1.0000**
0.24	0.6570	0.6694	1.3168	5.7775	1.1245	**1.0000**
0.28	0.9595	0.8142	1.3821	7.1839	1.1496	**1.0000**
0.32	1.6874	1.1064	1.5344	10.4657	1.2079	**1.0000**
0.36	5.8092	3.6510	2.3761	28.6232	1.5295	**1.0000**
0.40	4.5327	2.3891	0.2815	16.5469	0.7304	**1.0000**
0.44	1.7035	0.7451	0.8621	4.0262	0.9522	**1.0000**
0.48	1.0770	0.3915	0.9962	1.1399	1.0037	**1.0000**
0.52	0.8022	0.3156	1.0592	0.2173	1.0280	**1.0000**
0.56	0.6482	0.6685	1.0982	1.0561	1.0432	**1.0000**
0.60	0.5498	0.9899	1.1264	1.6607	1.0544	**1.0000**
0.64	0.4817	1.0554	1.1492	2.1422	1.0632	**1.0000**
0.68	0.4317	0.8259	1.1683	2.5524	1.0710	**1.0000**
0.72	0.3937	0.4354	1.1857	2.9187	1.0777	**1.0000**
0.76	0.3638	0.2889	1.2018	3.2564	1.0838	**1.0000**
0.80	0.3397	0.7917	1.2168	3.5750	1.0900	**1.0000**
0.84	0.3200	1.5726	1.2312	3.8796	1.0959	**1.0000**
0.88	0.3038	2.2219	1.2450	4.1749	1.1015	**1.0000**
0.92	0.2902	2.3909	1.2583	4.4619	1.1065	**1.0000**
0.96	0.2789	1.7639	1.2714	4.7430	1.1116	**1.0000**
MPE	0.0079	0.3035 *1.0e-004	0.1621 *1.0e-008	0.6020 *1.0e-008	0.1421 *1.0e-007	**0.1290 *1.0e-010**
RMS Error	0.0053	1.1167 *1.0e-005	1.017 *1.0e-009	3.3891 *1.0e-009	9.1330 *1.0e-010	**8.4323 *1.0e-010**

Table 10.6.2 provides point-wise errors over the 26 uniform mesh points for the methods "*FDM, Shooting Method, OCM based on Gaussian Knots, OCM based on* $a_1 = -0.57843961$ (*a best choice for Examples* 1-2), *OCM based on* $a_3 = -0.57737961$ (a best choice for Example 3) *and OCM based on an optimal choice of a**" when applied to Example 4

Table 10.6.2: Point-wise errors for Example 4 [25 Subintervals]

x_i =	FDM	Shooting Method 1.0e-004 *	OCM Gaussian Knots 1.0e-008 *	OCM *OZI-Knots* $a_1 = -0.57843961$ 1.0e-008 *	OCM *OZI-Knots* $a_3 = -0.57737961$ 1.0e-008 *	OCM with $a^* = -0.577399$ 1.0e-008 *
0.00	0	0	0	0	0	0
0.04	0.0014	0.0968	0.0742	0.2236	0.0661	0.0609
0.08	0.0027	0.2450	0.1247	0.3924	0.1107	0.1016
0.12	0.0039	0.3035	0.1532	0.5091	0.1353	0.1236
0.16	0.0050	0.2277	0.1621	0.5773	0.1421	0.1290
0.20	0.0059	0.0621	0.1546	0.6020	0.1341	0.1208
0.24	0.0067	0.0682	0.1342	0.5889	0.1146	0.1019
0.28	0.0073	0.0617	0.1047	0.5444	0.0871	0.0758
0.32	0.0077	0.0502	0.0696	0.4750	0.0548	0.0454
0.36	0.0079	0.0494	0.0322	0.3875	0.0207	0.0135
0.40	0.0079	0.0416	0.0049	0.2884	0.0127	0.0174
0.44	0.0078	0.0340	0.0394	0.1840	0.0435	0.0457
0.48	0.0075	0.0274	0.0697	0.0797	0.0702	0.0699
0.52	0.0072	0.0282	0.0946	0.0194	0.0918	0.0893
0.56	0.0067	0.0692	0.1136	0.1092	0.1079	0.1034
0.60	0.0062	0.1111	0.1265	0.1864	0.1184	0.1123
0.64	0.0056	0.1224	0.1333	0.2484	0.1233	0.1160
0.68	0.0050	0.0949	0.1343	0.2933	0.1231	0.1149
0.72	0.0043	0.0477	0.1300	0.3200	0.1182	0.1096
0.76	0.0037	0.0291	0.1210	0.3279	0.1091	0.1007
0.80	0.0030	0.0702	0.1079	0.3169	0.0966	0.0886
0.84	0.0024	0.1165	0.0912	0.2874	0.0812	0.0741
0.88	0.0017	0.1277	0.0716	0.2399	0.0633	0.0575
0.92	0.0011	0.0940	0.0495	0.1755	0.0435	0.0393
0.96	0.0006	0.0354	0.0255	0.0951	0.0223	0.0201
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
MPE	0.0079	0.3035 *1.0e-004	0.1621 *1.0e-008	0.6020 *1.0e-008	0.1421 *1.0e-007	**0.1290 *1.0e-010**
RMS Error	0.0053	1.1167 *1.0e-005	1.017 *1.0e-009	3.3891 *1.0e-009	9.1330 *1.0e-010	**8.4323 *1.0e-010**

Figure 10.6.1 **(Example 4)**

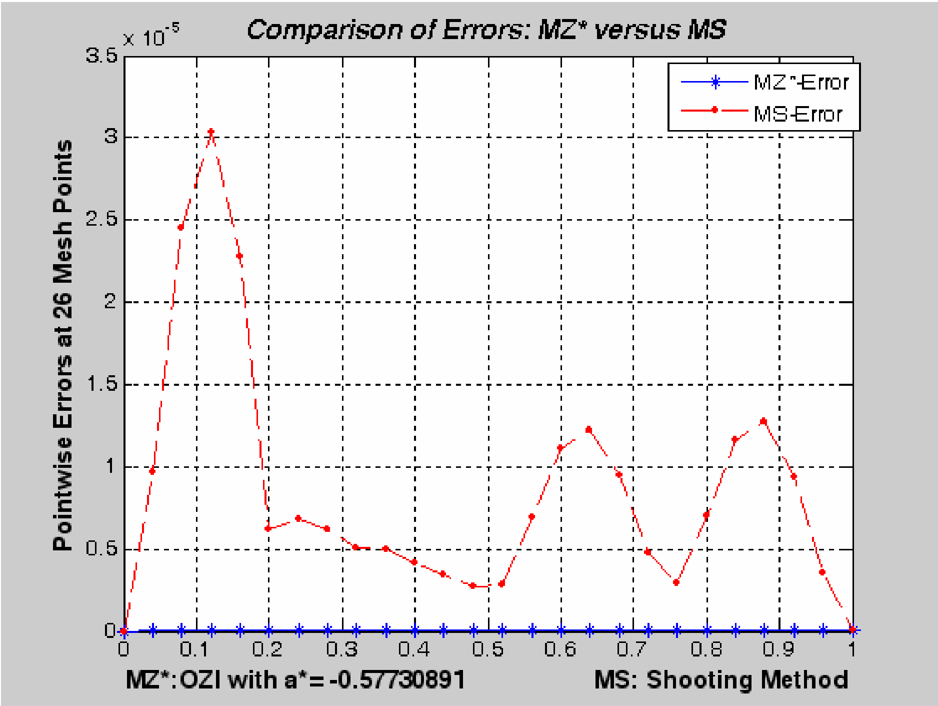


Figure 10.6.2 **(Example 4)**

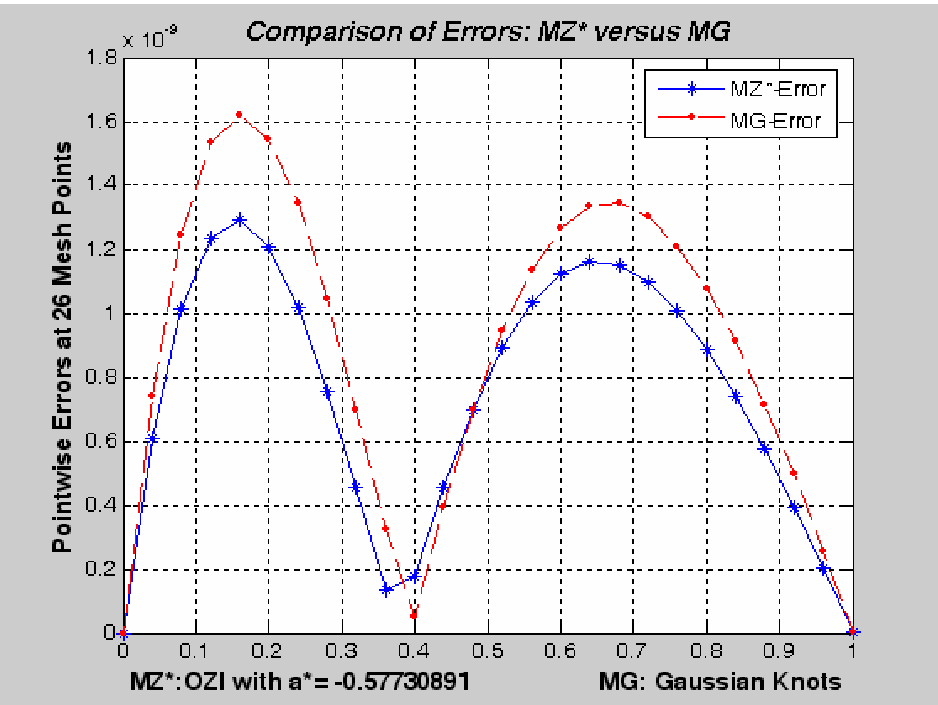


Figure 10.6.3 (Example 4)

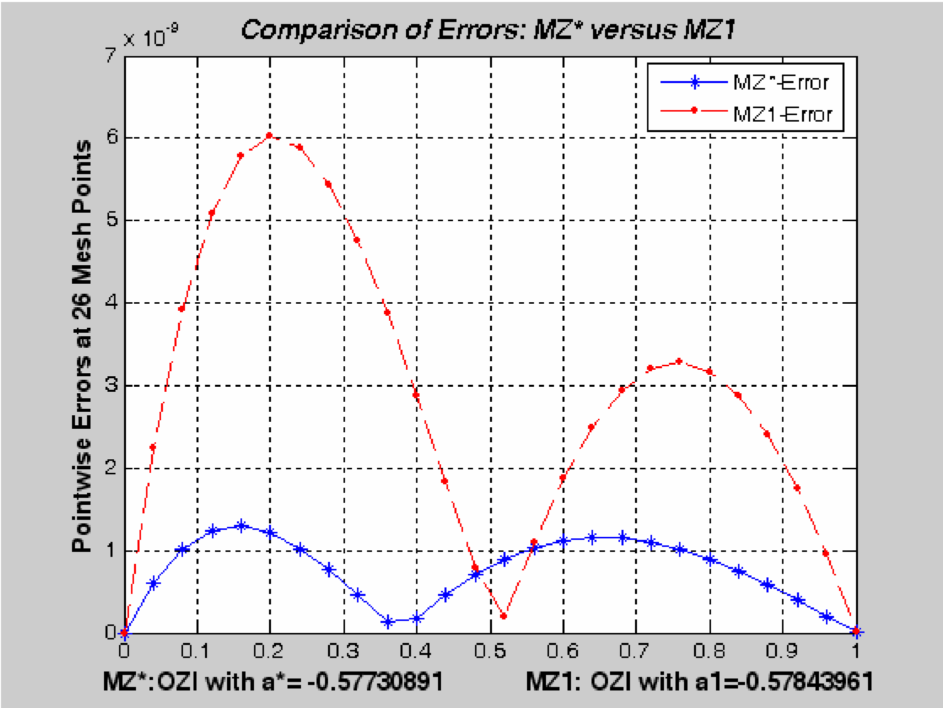
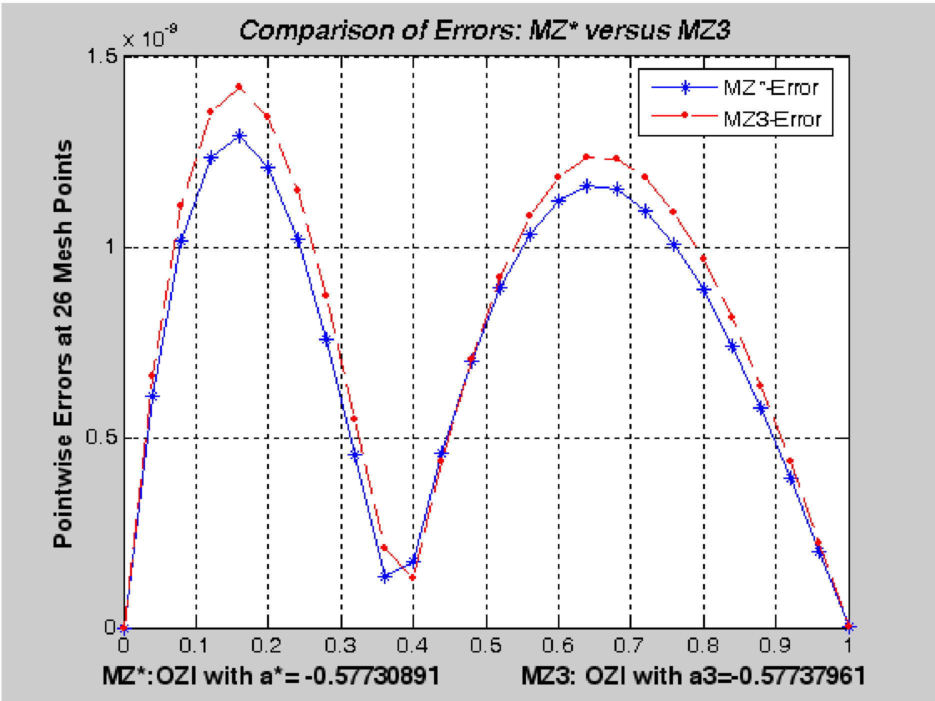


Figure 10.6.4 (Example 4)



10.7 Simulation results for Example 5

(BVP) $$y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 1 + x^2; \qquad y(0) = y(b) = 0$$

(Solution) $$y(x) = -\frac{b^3 + 3b}{6}x + \frac{x^4}{6} + \frac{x^2}{2}$$

Table 10.7.1 provides Comparative Accuracy when OCM based on an optimal choice $a^* = -0.577308910998$ is compared separately with the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots*, *OCM based on* $a_1 = -0.57843961$ (*a best choice for Examples* 1-2), $a_3 = -0.57737961$ (a best choice for Example 3), $a_4 = - -0.577399$ (a best choice for Example 4) for Example 5.

Table 10.7.1: Magnitude of $C_A(x_i)$ w.r.t. a^* [25 Subintervals]

x_i =	FDM 1.0e+009*	Shooting Method 1.0e+006*	OCM Gaussian Knots	OCM with a_1= −0.57843961 1.0e+004 *	OCM with a_3= −0.57737961	OCM with a_4= −0.577399 1.0e+004 *	OCM with a^*= −0.577308911
0.04	0.0917	0.3727	13.4204	0.0341	22.2435	0.0281	**1.0000**
0.08	0.1016	0.5367	14.4048	0.0368	23.9278	0.0302	**1.0000**
0.12	0.1130	0.5361	15.6275	0.0401	26.0198	0.0329	**1.0000**
0.16	0.1268	0.4049	17.1741	0.0443	28.6660	0.0363	**1.0000**
0.20	0.1438	0.2042	19.1767	0.0498	32.0927	0.0406	**1.0000**
0.24	0.1658	0.0499	21.8531	0.0572	36.6722	0.0465	**1.0000**
0.28	0.1955	0.1241	25.5876	0.0674	43.0623	0.0546	**1.0000**
0.32	0.2388	0.3750	31.1390	0.0826	52.5601	0.0667	**1.0000**
0.36	0.3082	0.6522	40.1935	0.1074	68.0502	0.0865	**1.0000**
0.40	0.4392	0.8565	57.4795	0.1547	97.6239	0.1242	**1.0000**
0.44	0.7823	0.9374	103.0404	0.2794	175.5759	0.2236	**1.0000**
0.48	4.1944	1.5762	557.5325	1.5236	953.2165	1.2151	**1.0000**
0.52	1.1736	0.2889	157.8304	0.4348	270.7747	0.3455	**1.0000**
0.56	0.4981	0.1634	67.9224	0.1886	116.9349	0.1494	**1.0000**
0.60	0.3082	0.0799	42.7063	0.1196	73.7882	0.0943	**1.0000**
0.64	0.2201	0.0512	31.0337	0.0876	53.8176	0.0689	**1.0000**
0.68	0.1686	0.0589	24.2420	0.0690	42.1955	0.0541	**1.0000**
0.72	0.1353	0.0405	19.8481	0.0570	34.6764	0.0445	**1.0000**
0.76	0.1117	0.0324	16.7445	0.0485	29.3641	0.0377	**1.0000**
0.80	0.0943	0.1488	14.4506	0.0422	25.4375	0.0327	**1.0000**
0.84	0.0810	0.2947	12.6923	0.0374	22.4279	0.0289	**1.0000**
0.88	0.0706	0.3837	11.3021	0.0336	20.0484	0.0258	**1.0000**
0.92	0.0622	0.3755	10.1779	0.0305	18.1247	0.0234	**1.0000**
0.96	0.0553	0.2536	9.2488	0.0280	16.5346	0.0214	**1.0000**
MPE	0.0222	0.4432 *1.0e-004	0.2959 *1.0e-008	0.8151 *1.0e-008	0.5077 *1.0e-008	0.6478 *1.0e-008	**0.1313 *1.0e-008**
RMS Error	0.0155	2.4414 *1.0e-005	2.1224 *1.0e-009	*1.0e-009	3.6390 *1.0e-009	4.6426 *1.0e-009	**8.1998 *1.0e-011**

Table 10.7.2 provides point-wise errors over the 26 uniform mesh points for the methods "*FDM*, *Shooting Method*, *OCM based on Gaussian Knots, OCM based on* $a_1 = -0.57843961$ (*a best choice for Examples* 1-2), *OCM based on* $a_3 = -0.57737961$ (*a best choice for Example* 3), $a_4 = -0.577399$ (*a best choice for Example* 4) *and OCM based on an optimal choice of a**" when applied to Example 5

Table 10.7.2: Point-wise errors [25 Subintervals]

x_i =	FDM	Shooting Method 1.0e-004 *	OCM Gaussian Knots 1.0e-008 *	OCM with a_1= −0.57843961 1.0e-008 *	OCM with a_3= −0.57737961 1.0e-008 *	OCM with a_4= −0.577399 1.0e-008 *	OCM with a^*= -0.577308911 1.0e-009 *
0.00	0	0	0	0	0	0	0
0.04	0.0031	0.1264	0.0455	0.1155	0.0754	0.0952	0.0339
0.08	0.0061	0.3250	0.0872	0.2226	0.1449	0.1831	0.0606
0.12	0.0091	0.4294	0.1252	0.3213	0.2084	0.2635	0.0801
0.16	0.0118	0.3756	0.1593	0.4113	0.2659	0.3364	0.0928
0.20	0.0142	0.2020	0.1896	0.4928	0.3174	0.4019	0.0989
0.24	0.0164	0.0494	0.2162	0.5655	0.3628	0.4598	0.0989
0.28	0.0183	0.1159	0.2390	0.6293	0.4022	0.5102	0.0934
0.32	0.0198	0.3107	0.2579	0.6840	0.4354	0.5528	0.0828
0.36	0.0209	0.4432	0.2731	0.7295	0.4624	0.5877	0.0680
0.40	0.0217	0.4240	0.2845	0.7656	0.4832	0.6148	0.0495
0.44	0.0222	0.2657	0.2921	0.7921	0.4978	0.6339	0.0283
0.48	0.0223	0.0837	0.2959	0.8087	0.5059	0.6449	0.0053
0.52	0.0220	0.0542	0.2959	0.8151	0.5077	0.6478	0.0187
0.56	0.0214	0.0703	0.2921	0.8112	0.5029	0.6424	0.0430
0.60	0.0205	0.0532	0.2845	0.7966	0.4916	0.6286	0.0666
0.64	0.0194	0.0450	0.2731	0.7711	0.4737	0.6063	0.0880
0.68	0.0179	0.0626	0.2580	0.7344	0.4490	0.5753	0.1064
0.72	0.0163	0.0487	0.2390	0.6862	0.4176	0.5356	0.1204
0.76	0.0144	0.0418	0.2162	0.6261	0.3792	0.4870	0.1291
0.80	0.0124	0.1953	0.1897	0.5539	0.3339	0.4293	0.1313
0.84	0.0102	0.3700	0.1593	0.4693	0.2816	0.3624	0.1255
0.88	0.0078	0.4250	0.1252	0.3719	0.2221	0.2862	0.1108
0.92	0.0053	0.3220	0.0873	0.2615	0.1554	0.2004	0.0857
0.96	0.0027	0.1248	0.0455	0.1376	0.0814	0.1051	0.0492
1.00	0.0000	0.0000	0.0000	0.0000	0	0.0000	0.0000
MPE	0.0222	0.4432 *1.0e-004	0.2959 *1.0e-008	0.8151 *1.0e-008	0.5077 *1.0e-008	0.6478 *1.0e-008	**0.1313 *1.0e-008**
RMS Error	0.0155	2.4414 *1.0e-005	2.1224 *1.0e-009	*1.0e-009	3.6390 *1.0e-009	4.6426 *1.0e-009	**8.1998 *1.0e-011**

Fig 10.7.1: **(Example 5)**

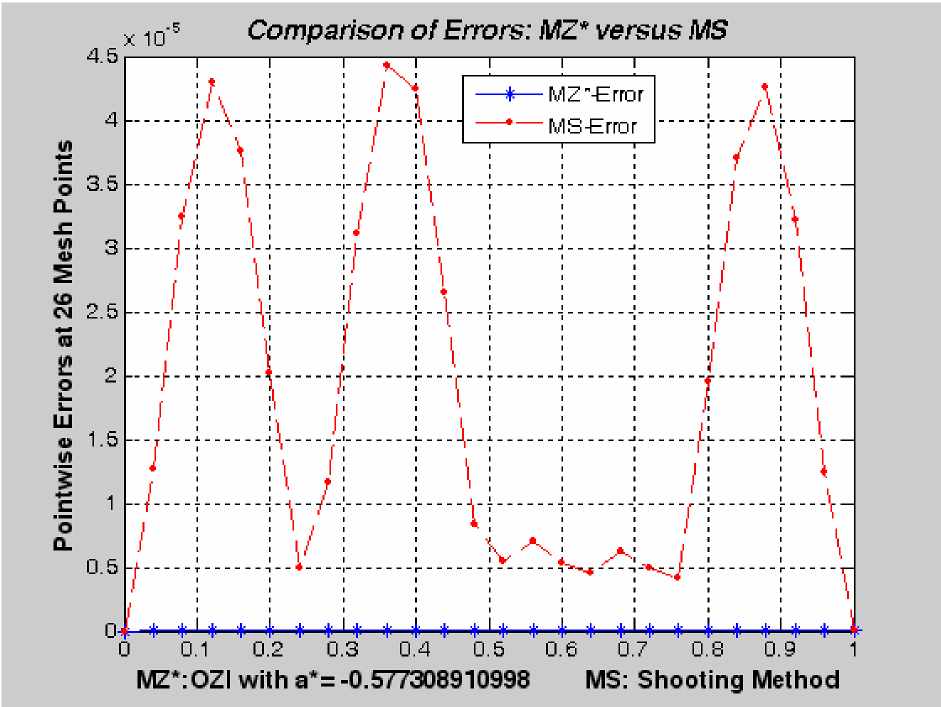


Fig 10.7.2: **(Example 5)**

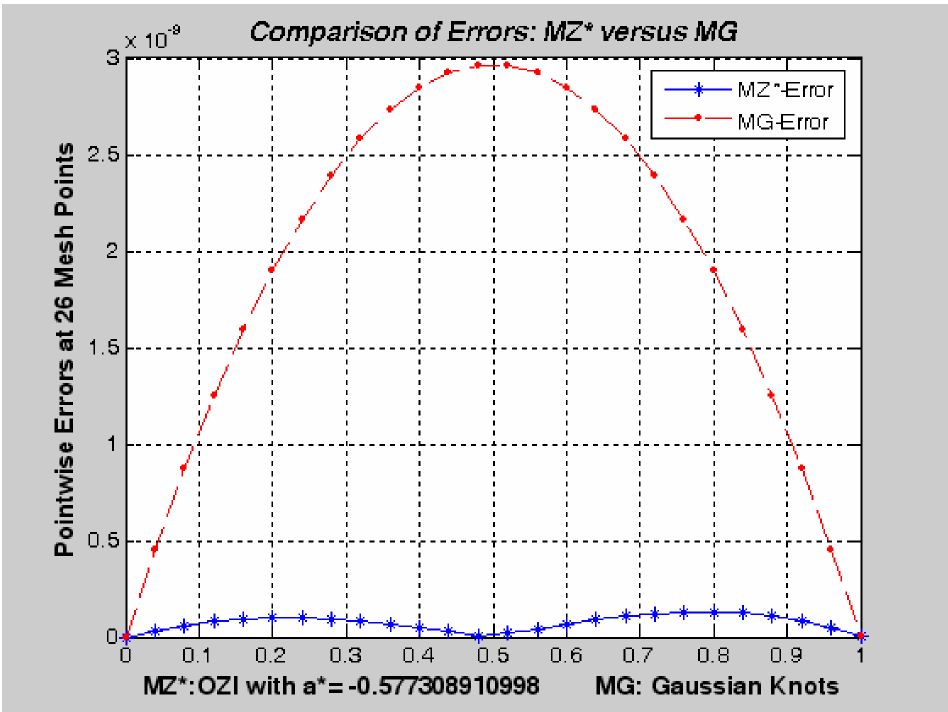


Fig 10.7.3: **(Example 5)**

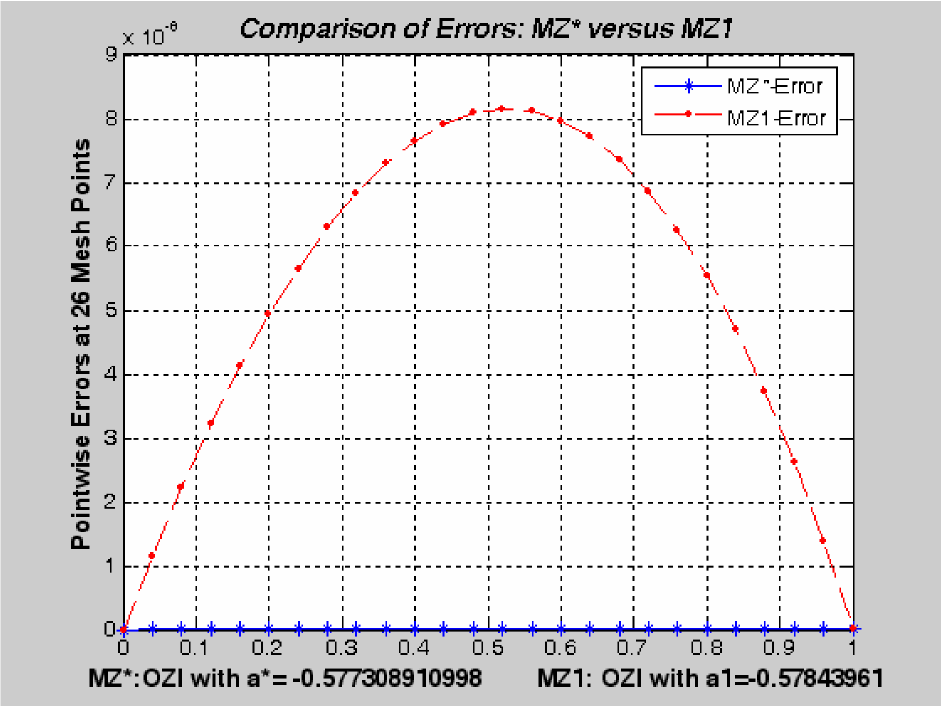


Fig 10.7.4: **(Example 5)**

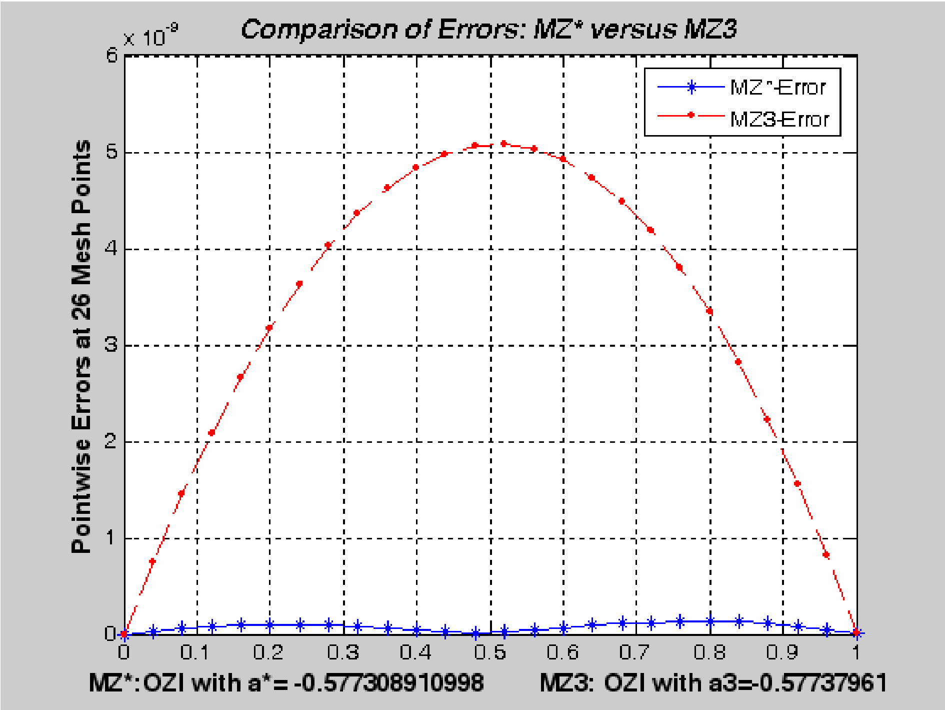
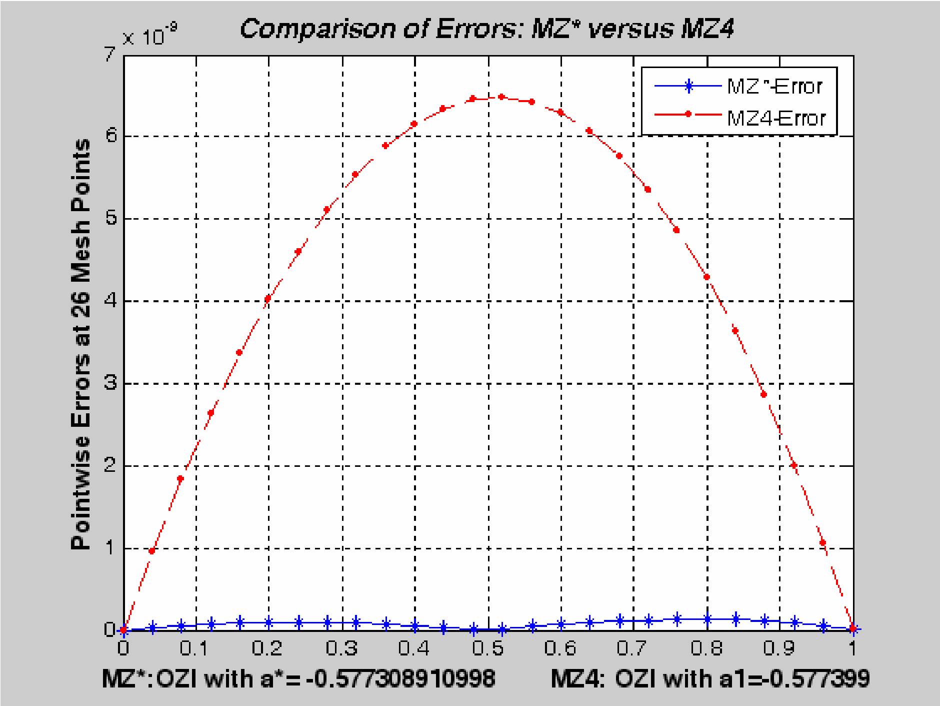


Fig 10.7.5: (Example 5)



10.8 Inefficiency of OCM and other methods

In general, every approximation solution of a mathematical model has some limitations. We had similar observations in terms of inefficiency of the approximate methods when applied to the BVP's subject to an oscillatory forcing function We discuss here an example where FDM and OCM eith based on Gaussian Knots or on an orthogonal pair do not provide promising results when the forcing function is oscillatory [43].

Example 6

$$y'' = -f\,; \qquad y(-1) = y(1) = 0 \tag{10.4}$$

The exact solution of the BVP (10.4) when $f = (m\pi)^2 \sin(m\pi x)$ is given by $y = \sin(m\pi x)$. We present the comparison of approximate methods for m = 1, 5 in the following tables:

Table 10.8.1: Point-wise errors for Example 6 [m = 1, # of mesh points = 26]

x_i =	FDM	Shooting Method 1.0e-004 *	OCM Gaussian Knots 1.0e-005 *	OCM with a^* = - 0.5779573961 1.0e-006 *
0.00	0.0000	0.0000	0.0000	0.0000
0.04	0.0013	0.0000	0.0691	0.0089
0.08	0.0025	0.2224	0.1339	0.0256
0.12	0.0036	0.1122	0.1903	0.0493
0.16	0.0045	0.6202	0.2347	0.0782
0.20	0.0050	0.1822	0.2644	0.1107
0.24	0.0053	0.1782	0.2775	0.1447
0.28	0.0052	0.1497	0.2731	0.1780
0.32	0.0048	0.2386	0.2515	0.2086
0.36	0.0041	0.3597	0.2142	0.2345
0.40	0.0031	0.3693	0.1634	0.2542
0.44	0.0019	0.0089	0.1023	0.2663
0.48	0.0007	0.0003	0.0348	0.2701
0.52	0.0007	0.0003	0.0348	0.2655
0.56	0.0019	0.0089	0.1023	0.2525
0.60	0.0031	0.3693	0.1634	0.2322
0.64	0.0041	0.3597	0.2142	0.2057
0.68	0.0048	0.2386	0.2515	0.1747
0.72	0.0052	0.1497	0.2731	0.1412
0.76	0.0053	0.1782	0.2775	0.1073
0.80	0.0050	0.1822	0.2644	0.0751
0.84	0.0045	0.6202	0.2347	0.0466
0.88	0.0036	0.1122	0.1903	0.0236
0.92	0.0025	0.2224	0.1339	0.0076
0.96	0.0013	0.0000	0.0691	0.0004
1.00	0.0000	0.0000	0.0000	0.0000
MPE	0.0053	0.6202 *1.0e-004	0.2775 *1.0e-005	**0.2701 *1.0e-006**
RMS Error	0.0037	2.5674 *1.0e-005	1.9276 *1.0e-006	**1.6197 *1.0e-07**

Table 10.8.2: Point-wise errors for Example 6 [m = 5, # of mesh points = 26]

x_i =	FDM	Shooting Method 1.0e-003 *	OCM Gaussian Knots	OCM with a^*= - 0.5779573961
0.00	0.0000	0.0000	0.0000	0.0000
0.04	0.1357	0.0776	0.0018	0.0017
0.08	0.0839	0.0244	0.0011	0.0011
0.12	0.0839	0.1428	0.0011	0.0010
0.16	0.1357	0.1913	0.0018	0.0017
0.20	0.0000	0.0140	0.0000	0.0000
0.24	0.1357	0.1386	0.0018	0.0017
0.28	0.0839	0.3408	0.0011	0.0011
0.32	0.0839	0.2732	0.0011	0.0010
0.36	0.1357	0.1113	0.0018	0.0017
0.40	0.0000	0.0944	0.0000	0.0000
0.44	0.1357	0.0465	0.0018	0.0017
0.48	0.0839	0.0229	0.0011	0.0011
0.52	0.0839	0.0229	0.0011	0.0010
0.56	0.1357	0.0465	0.0018	0.0017
0.60	0.0000	0.0944	0.0000	0.0000
0.64	0.1357	0.1113	0.0018	0.0017
0.68	0.0839	0.2732	0.0011	0.0011
0.72	0.0839	0.3408	0.0011	0.0010
0.76	0.1357	0.1386	0.0018	0.0017
0.80	0.0000	0.0140	0.0000	0.0000
0.84	0.1357	0.1913	0.0018	0.0017
0.88	0.0839	0.1428	0.0011	0.0011
0.92	0.0839	0.0244	0.0011	0.0010
0.96	0.1357	0.0776	0.0018	0.0017
1.00	0.0000	0.0000	0.0000	0.0000
MPE	0.1357	0.3408 *1.0e-003	0. 0018	**0.** 0017
RMS Error	2.0303	1.5135 *1.0e-004	0.0013	**0.0013**

Figure 10.8.1: ***m*** **= 1 (Example 6)**

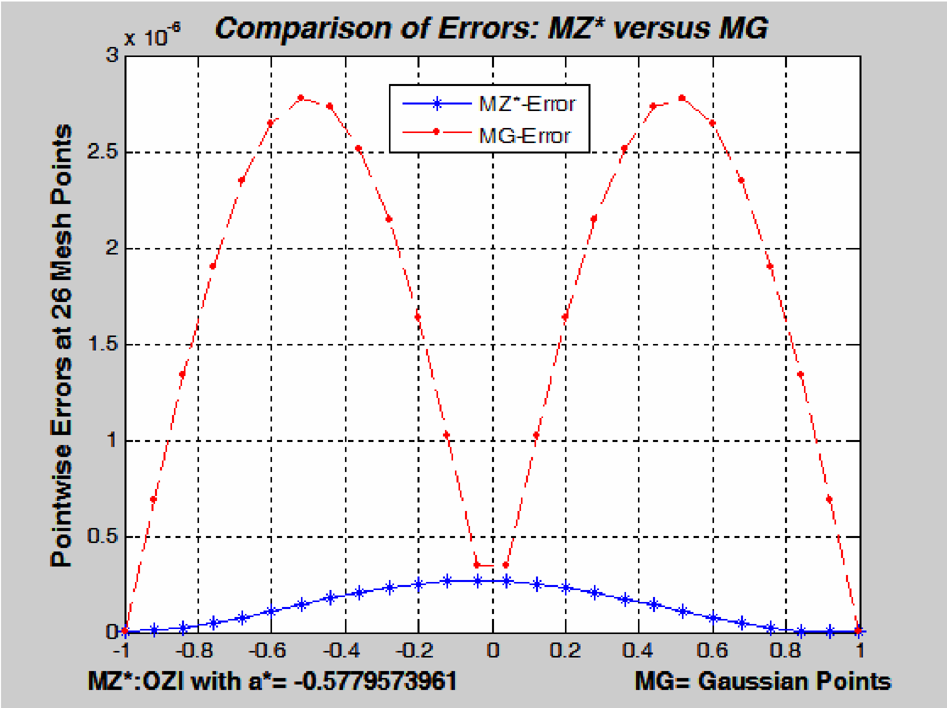
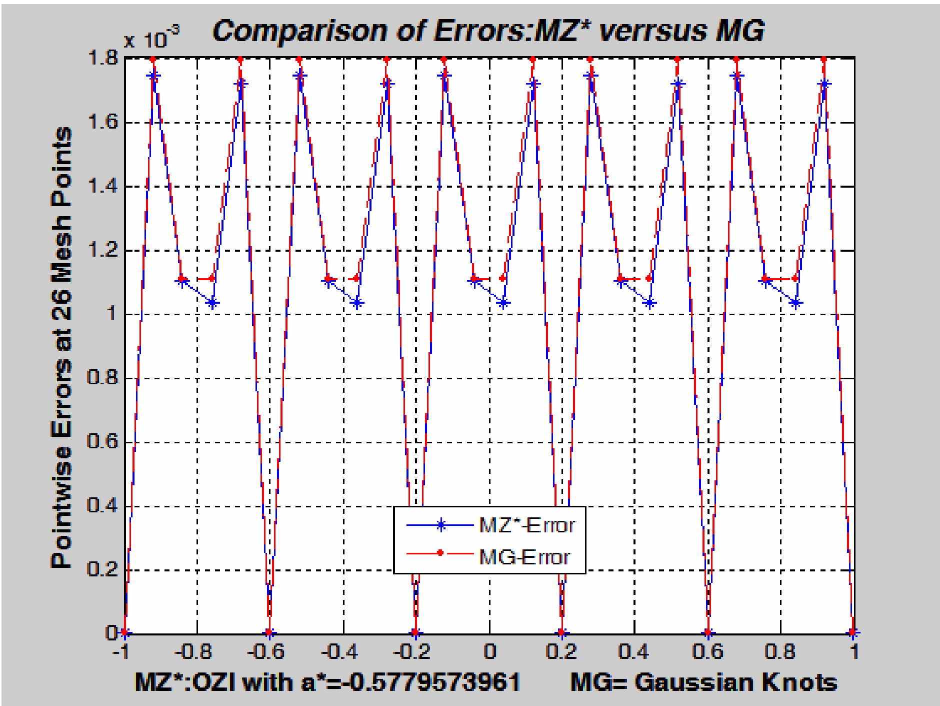


Figure 10.8.2: ***m*** **= 5 (Example 6)**



Remark 10.1. The above observations indicate that the OCM based on an orthogonal pair with a^*= - 0.5779573961 performs well as compared to the OCM based on the Gaussian points when the forcing function is not oscillatory in the underlying interval [–1,1], i.e. the case m =1 (Table 10.8.1 and Figure 10.8.1). In this case, the error of approximation is even far better to that obtained by the finite difference method and linear shooting method.

On the other hand, with the increase of oscillations in the forcing function in [–1,1], OCM in both cases show a poor performance even worse to the shooting method, i.e. the case m =5 (Table 10.8.2 and Figure 10.8.2).

McBain and Armfield have proposed a procedure [43] based on the Green's function and the product integration. Their approach provides a better approximation when the forcing function suffers severe oscillations in the underlying interval of a BVP.

11. Further Directions of Work

In this chapter we identify some directions based on our work. The related problems which involve both theoretical as well as computational aspects are narrated below

11.1 Problem A: *Use of weighted orthogonal zeros*

In the process of applying OCM, we may determine a pair of weighted zeros by considering a quadratic polynomial of the form $(t-t^*)^2$ as an underlying weight function. Here, in fact, we fix a real number t^* and construct a 2nd degree orthogonal polynomial $q_{i,2}$ over the interval $[x_i, x_{i+1}]$ with respect to the weight function $w_i(t) = (t-t_i^*)^2$. As noted earlier, $q_{i,2}$ can be constructed by the 3-term recurrence relation and its zero can be found by the following procedure:

i. Set $q_{i,0}(t) = 1$ and $q_{i,1}(t) = (t-\alpha_{i,1})q_{i,0}(t)$ were $\alpha_{i,1} = \dfrac{\langle tq_{i,1}, q_{i,1}\rangle}{\langle q_{i,1}, q_{i,1}\rangle}$. In this case, the notation $\langle .\,,.\rangle$ stands for the inner product defined as $\langle h, g\rangle := \int_{x_i}^{x_{i+1}} h(t)g(t)w_i(t)dt$.

ii. Set $q_{i,2}(t) = (t-\alpha_{i,2})q_{i,2}(t) - \beta_{i,2}q_{i,0}(t)$ where

$$\alpha_{i,2} = \frac{\langle tq_{i,2}, q_{i,2}\rangle}{\langle q_{i,2}, q_{i,2}\rangle} \text{ and } \beta_{i,2} = \frac{\langle q_{i,2}, q_{i,2}\rangle}{\langle q_{i,1}, q_{i,1}\rangle}.$$

iii. The two zeros of $q_{i,2}$ are the eigenvalues λ_1 and λ_2 of the matrix

$$A_i = \begin{bmatrix} \alpha_{i,1} & \sqrt{\beta_{i,2}} \\ \sqrt{\beta_{i,2}} & \alpha_{i,2} \end{bmatrix}.$$

We shall denote these zeros by $z_{i,1}$ and $z_{i,2}$ i.e., $z_{i,1} = \lambda_1$ and $z_{i,2} = \lambda_2$, and refer them to as the two orthogonal weighted zeros (*OWZ*) of $p_{i,2}$.

An investigation of the consequence of OWZ on the approximate solution of the BVP via OCM may provide interesting observation.

11.2 Problem B: *Use of orthogonal triplets*

While approximating the solution of two-point BVP, one may think to elaborate on the method of orthogonal collocation and expect an improvement in the order of convergence. In this regard, three knots (either shifted zeros of 3rd degree Legendre polynomial or orthogonal triplet) are shifted to each of subintervals which are support of certain piecewise quartic Hermite polynomials. Construction of the points is carried out on the following line:

A. Three Legendre zeros

i. The 3rd degree Legendre polynomial has three zeros: $t_1 = -\frac{3}{\sqrt{5}},\ t_2 = 0,\ t_2 = \frac{3}{\sqrt{5}}$

ii. For any interval $[x_i, x_{i+1}]$, set

$$z_{i,j} = x_i + \frac{h_i}{2}(1+t_j). \qquad j = 1,2,3.$$

where $h_i = x_{i+1} - x_i$

iii. The points $z_{i,j}, j = 1,2,3$ are termed as three Gaussian points in the interval $[x_i, x_{i+1}]$. If we set $p_{i,3}(t) = (t - z_{i,1})(t - z_{i,2})(t - z_{i,3})$, then

$$\int_{x_i}^{x_{i+1}} p(t)\, p_{i,3}(t)dt = 0, \qquad \forall p \in \pi_2 .$$

B. Orthogonal Triplet

i. Here, we determine 3rd degree *OZI* (cf (5.2)) and its three zeros in [– 1, 1]. Taking into account the steps (i) and (ii) of Section 5.3, we set (using 3-term recurrence relation) $\varphi_3(t) = (t - \alpha_2)\varphi_2(t) - \beta_2\varphi_1(t)$ where

$$\alpha_2 = \frac{\langle t\varphi_2, \varphi_2\rangle}{\langle \varphi_2, \varphi_2\rangle} \text{ and } \beta_2 = \frac{\langle t\varphi_2, \varphi_2\rangle}{\langle \varphi_2, \varphi_2\rangle} .$$

ii. Note that $b_1 = a_1$ is one of the three zeros of φ_3. Its remaining two zeros are determined by following Golub and Welsch's method, i.e.,

a. Set $A = \begin{bmatrix} \alpha_1 & \sqrt{\beta_2} \\ \sqrt{\beta_2} & \alpha_2 \end{bmatrix}$

b. The matrix A has two real and distinct eigenvalues, say λ_1 and λ_2.

c. $b_2 = \lambda_1$ and $b_3 = \lambda_2$ will be the remaing two zeros of φ_3.

Definition 11.1. We shall call the points (b_1, b_2, b_3) (cf (ii) in 5.2) an orthogonal triplet of points in [– 1, 1].

Definition 11.2. Let (b_1, b_2, b_3) be an orthogonal triplet of zeros in [– 1, 1]. For any interval $[x_i, x_{i+1}]$, set

$$z_{i,1} = x_i + \frac{h_i}{2}(1+b_1),\quad z_{i,2} = x_i + \frac{h_i}{2}(1+b_2),\quad z_{i,3} = x_i + \frac{h_i}{2}(1+b_3)$$

where $h_i = x_{i+1} - x_i$ The triplet of points $(z_{i,1}, z_{i,2}, z_{i,3})$ will be termed as "triplet of *OZI* points" in the interval $[x_i, x_{i+1}]$ corresponding to orthogonal triplet (b_1,b_2,b_3).

11.3 Piecewise quartic Hermite polynomial

The quartic Hermite polynomial for a function $f : [t_1, t_2] \to \Re$ is a 4th degree polynomial *p* which satisfies the four constraints similar to those given for cubic Hermite polynomial (cf (5.2)). Since quartic polynomial involves five parameters, we have to add an additional constraint in order to define *p*. For this, a natural choice is to select a constraint in the form of 2nd derivative. Also, each of the fundamental polynomials will require five conditions. For the sake of symmetry, we increase the smoothness of $\phi_1, \phi_2, \psi_1, \psi_2$ at the points where these polynomials have 0-value and 0-derivative (cf (5.4)). The five pieces of quartic polynomials in order to furnish the desired basis will satisfy the following conditions:

$$\left.\begin{array}{l}\left.\begin{array}{ll}\phi_i(t_j)=\delta_{ij} & \psi_i(t_j)=0\\ \phi_i'(t_j)=0 & \psi_i'(t_j)=\delta_{ij}\\ \gamma(t_k)=0 & \gamma'(t_k)=0\end{array}\right\},\quad i,j,k=1,2\\ \begin{array}{ll}\phi_1''(t_1)=0 & \psi_1''(t_2)=0\\ \phi_2''(t_2)=0 & \psi_2''(t_1)=0\\ \gamma''(t_1)=c \;\text{ or } & \gamma''(t_2)=c\end{array}\end{array}\right\}. \tag{5.11}$$

where $c > 0$. Fixing $c > 0$ and another constant C, the polynomial

$$p(x) = f(t_1)\phi_1(x) + f(t_2)\phi_2(x) + f'(t_1)\psi_1(x) + f'(t_2)\psi_2(x) + C\gamma(x) \tag{5.12}$$

is uniquely defined and satisfies the conditions (5.4). Now we define five types of quartic polynomials: $\phi_{L,i}$, $\phi_{R,i}$, $\psi_{U,i}$, $\psi_{L,i}$, γ on each subinterval $[x_i, x_{i+1}]$ based on the following properties:

$$\left.\begin{array}{lllll}\phi_{L,i}(x_i) & =\phi_{L,i}'(x_i) & =\phi_{L,i}''(x_i) & =\phi_{L,i}'(x_{i+1}) & =0, \quad \phi_{L,i}(x_{i+1})=1\\ \phi_{R,i}(x_{i+1}) & =\phi_{R,i}'(x_{i+1}) & =\phi_{R,i}''(x_{i+1}) & =\phi_{R,i}'(x_i) & =0, \quad \phi_{R,i}(x_i)=1\\ \psi_{U,i}(x_{i+1}) & =\psi_{U,i}'(x_{i+1}) & =\psi_{U,i}''(x_{i+1}) & =\psi_{U,i}(x_i) & =0, \quad \psi_{U,i}'(x_i)=1\\ \psi_{L,i}(x_{i+1}) & =\psi_{L,i}'(x_{i+1}) & =\psi_{L,i}''(x_{i+1}) & =\psi_{L,i}(x_i) & =0, \quad \psi_{L,i}'(x_{i+1})=1\\ \gamma(x_i) & =\gamma'(x_i) & =\gamma(x_{i+1}) & =\gamma'(x_{i+1}) & =0, \quad \gamma''(x_i)=1\end{array}\right\} \tag{5.13}$$

An explicit representation of each piece is is given below.

i. Left piece of ϕ: $\phi_{L,i}$

$$\phi_{L,i}(t)=\begin{cases}A_L(t-x_i)^4+B_L(t-x_i)^3(t-x_{i+1}), & t\in[x_i,x_{i+1}]\\ 0, & \text{else where}\end{cases} \tag{5.14}$$

with $A_L = \frac{1}{h^4}, B_L = \frac{-4}{h^4}$

ii. Right piece of ϕ: $\phi_{R,i}$

$$\phi_{R,i}(t)=\begin{cases}A_R(t-x_{i+1})^4+B_R(t-x_{i+1})^3(t-x_i), & t\in[x_i,x_{i+1}]\\ 0, & \text{else where}\end{cases} \tag{5.15}$$

with $A_L = \frac{1}{h^4}, B_L = \frac{-4}{h^4}$

iii. Upper piece of ψ : $\psi_{U,i}$

$$\psi_{U,i}(t) = \begin{cases} C_U (t - x_i)(t - x_{i+1})^3, & t \in [x_i, x_{i+1}] \\ 0, & \text{else where} \end{cases} \tag{5.16}$$

with $C_U = \dfrac{-1}{h^3}$

iv. Lower piece of ψ : $\psi_{L,i}$

$$\psi_{L,i}(t) = \begin{cases} C_L (t - x_i)^3 (t - x_{i+1}) & t \in [x_i, x_{i+1}] \\ 0, & \text{else where} \end{cases} \tag{5.17}$$

with $C_L = \dfrac{1}{h^3}$

v. Additional piece: γ_i

$$\gamma_i(t) = \begin{cases} D_g (t - x_i)^2 (t - x_{i+1})^2 & t \in [x_i, x_{i+1}] \\ 0, & \text{else where} \end{cases} \tag{5.18}$$

with $D_g = \dfrac{1}{2h^2}$

11.4 Proposed work

One may conduct a detailed study by using three knots (as explained in A and B above) with the piecewise quartic Hermite polynomials and then compare the resulting collocation solution with other methods.

12. Concluding Remarks

By defining a notion of orthogonal pair of points (a_1, a_2) in the interval [–1,1] with $a_1 \neq 0$, we have introduce an orthogonal collocation method (OCM) with adjustable collocation points. We refer to this method as the "OCM based on the orthogonal pair (a_1, a_2)". This method turns out the Gaussian collocation method as a special case when $a_1 = {-1}/{\sqrt{3}}$.

We have explained a simple procedure of defining four basic fundamental cubic and quartic Hermite polynomials (see Sections 6.2 and 11.3) which may be compared with explanations given in the standard texts, e.g., [42, 49,]. The pent-diagonal coefficient matrix which appears in the linear system as a consequence of OCM has a simple format when expressed with these fundamental functions (see (7.10)).

Approximate solutions of a two-point linear boundary value problem is discussed by different methods namely, finite difference method, linear shooting method and orthogonal collocation method. We noted that the former one being of order h^2 is inferior to the latter two.

We have established the $O(h^{3.5})$-convergence for OCM based on orthogonal pair (a_1, a_2) when a_1 is in the neighborhood of ${-1}/{\sqrt{3}}$. in the interval [–1,1]. As a corrolary to this result, we note that same order of convergence holds for OCM based on Gaussian points. This rectify the omission which appear in [49, page 308, (13)]

We can determine several orthogonal pairs (a_1, a_2) in [–1,1] with a_1 in the neighborhood of ${-1}/{\sqrt{3}}$ for which the OCM has a better performance when compared with OCM based on Gaussian points (see examples 1-6 in Chapter 10). This negates the concept that Gaussian points leads to optimal error when considered in OCM along with piecewise C-H polynomials.

The orthogonal collocation methods show a better performance in comparison with the linear shooting methods when the forcing function does not suffer oscillations in the underlying interval (see Examples 1, 3, 4 and 5 in Chapter 10). With an increase of oscillations in the forcing function, the linear shooting method does a better job (see Examples 2 and 6 in Chapter 8).

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14. Presentations & Publications

A part of work carried out in the project has been presented at the following forums:

1) International Conference on Industrial & Applied Mathematics (iciam, 07), Zurich, Switzerland (July, 2007).
2) MATH Seminars, Department of Mathematics & Statistics, KFUPM, (May, 2008)

We are also planning to present the entire outcome of the project in the "Conference Numerical in Analysis (NumAn 2008) - Recent Approaches to Numerical Analysis: Theory, Methods and Applications", Kalamata, Greece, Sept. 01-05, 2008.

The details of the publications out of the project report are as follows:

1. The manuscript "On orthogonal collocation by the zeros of orthogonal 0-interpolants" has been accepted for publication in PAMM, an on-line Journal of Wiley-VCH, and will appear in June, 2008.
2. The manuscript "On the Choice of Knots in Orthogonal Collocation Method" has been accepted for conference publication which will appear in "Lecture Notes in Computer Science (LNCS), Springer, (September, 2008).

Soft copies of both manuscripts and the final report are available on CD submitted to KFUPM Research Committee.