Section 14.8 *Maxima & minima of functions of two variables*

Learning outcomes

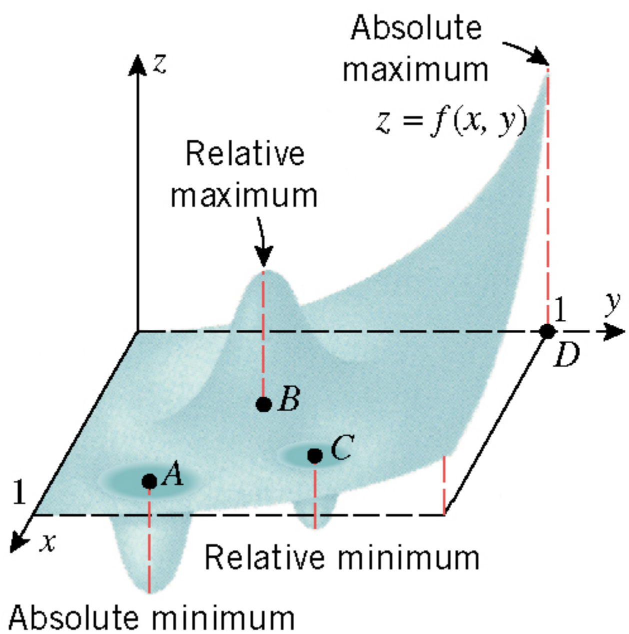
After completing this section, you will inshaAllah be able to

1. explain what is meant by relative maxima or relative minima
2. explain what is meant by absolute maxima or absolute minima
3. find relative extrema of $z=f(x,y)$
4. find absolute extrema of $z=f(x,y)$

What are different types of extrema

$f(x,y)$ has a relative maximum at (x_0,y_0) if $f(x,y) \geq f(x_0,y_0)$ for all points near (x_0,y_0)

$f(x,y)$ has a relative minimum at (x_0,y_0) if $f(x,y) \leq f(x_0,y_0)$ for all points near (x_0,y_0)



$f(x,y)$ has an absolute maximum at (x_0,y_0) if $f(x,y) \geq f(x_0,y_0)$ for all the points in domain of $f(x,y)$

$f(x,y)$ has an absolute minimum at (x_0,y_0) if $f(x,y) \leq f(x_0,y_0)$ for all the points in domain of $f(x,y)$

Main question:

How to find relative or absolute extrema?

Finding relative extrema of *z=f(x,y)*

STEP 1: Find critical points of $f(x,y)$

These are all possible candidates for extrema

A point (x_0, y_0) is a critical point of $f(x,y)$ if

- $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$

OR

- $f_x(x_0,y_0)$ and/or $f_y(x_0,y_0)$ do not exist.

i.e. all points where partial derivatives are zero or do not exist.

STEP 2: Determine whether or not $f(x,y)$ has extrema at critical points

Find $D = f_{xx}(x_0,y_0)\,f_{yy}(x_0,y_0) - f_{xy}{}^2(x_0,y_0)$

Second partials test

1. If $D>0$ and $f_{xx}(x_0,y_0)>0$ then (x_0,y_0) is a relative minimum.
2. If $D>0$ and $f_{xx}(x_0,y_0)<0$ then (x_0,y_0) is a relative maximum.
3. If $D<0$ then (x_0,y_0) is a saddle point.
4. If $D=0$ then no conclusion can be drawn and other techniques would be needed.

See class explanation

Example 14.8.1 Find, if any, the relative extrema or saddle points of $f(x,y)=4+x^3+y^3-3xy$.

Example 14.8.2 Show that the second partials test fails to find the extrema of $f(x,y)=x^4+y^4$. Use some other way to find its relative maxima and minima (if any).

Both Solutions Done in class.

Exercise Find, if any, the relative extrema or saddle points of $f(x,y)=3x^2y+y^3-3x^2-3y^2+2$. (Get answer in class)

Finding absolute extrema of $z=f(x,y)$ on closed & bounded sets

- Note: The existence of absolute extrema of $f(x,y)$ is guaranteed over closed & bounded domains.

Extreme value theorem

If $f(x,y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.

These absolute extrema can occur on the boundary of R or at an interior (critical) point

A set of points in 2-space is called

- closed if it contains all of its boundary points.
- bounded it can be contained in a rectangle or disk. In other words, it is a finite region.

See class explanation

Procedure for finding absolute extrema on closed & bounded set R

5. Find all critical points that lie in the interior of R.
    a. Then find the value of $f(x,y)$ at these points
6. Find all extrema of $f(x,y)$ at the boundary points.
    a. This will involve methods of extrema from Math101. (See examples)
7. Look at the values of $f(x,y)$ found in Step 1 & Step 2. The largest is absolute maximum & the smallest is absolute minimum.

Example 14.8.3 Find the absolute minimum and absolute maximum of $f(x,y)=x^2+4y^2-2x^2y+4$ on the rectangle $-1\le x\le 1, -1\le y\le 1$.

Solution Done in class.

Exercise Find the absolute minimum and absolute maximum of $f(x,y)=2x^2-y^2+6y$ on the region $x^2+y^2\le 16$.

Hint discussed in class What is the boundary of region $x^2+y^2\le 16$? How to use it in Step 2 of procedure?

Exercise (Applied problem)

A rectangular box with no top is to be constructed to have a volume $V=12\,ft^3$. The cost per square foot of the material to be used is SR.4 for the bottom, SR.3 for two of the opposite sides, and SR.2 for remaining pair of opposite sides. Find the dimensions of the box that will minimize the cost.

Hints

- Let x,y,z be sides. Then we have
  - area of base = xy
  - two sides of area = xz
  - two sides of area = yz
- Hence, the cost is given by $C=4xy+3(2xz)+2(2yz)$
- Using $xyz=12$ we can write C as function of two variables
  - i.e. $C(x,y)=4xy+\dfrac{72}{y}+\dfrac{48}{x}$ is to be minimized for $x>0,\ y>0$.
- Note $C(x,y)$ is to be minimized for $x>0,\ y>0$. That means it is not an extrema problem over closed & bounded region.
  - Hence, there is no guarantee of absolute minimum (by theory)
    - But from physical situation we have assurance of minimum.
- So we can continue looking for minimum using the method of relative extrema.
- Complete from here. Answer: $x=2, y=3, z=2$

Do Qs: 1-40 *End of 14.8*