Section 14.6 *Directional Derivatives and Gradients*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. explain what is meant by directional derivative
2. compute directional derivative
3. explain what is meant by gradient vector
4. know and apply important facts about gradient vectors

We have done partial derivatives

- f_x : rate of change of 'f' in x-direction
- f_y : rate of change of 'f' in y-direction

Here we do directional derivatives

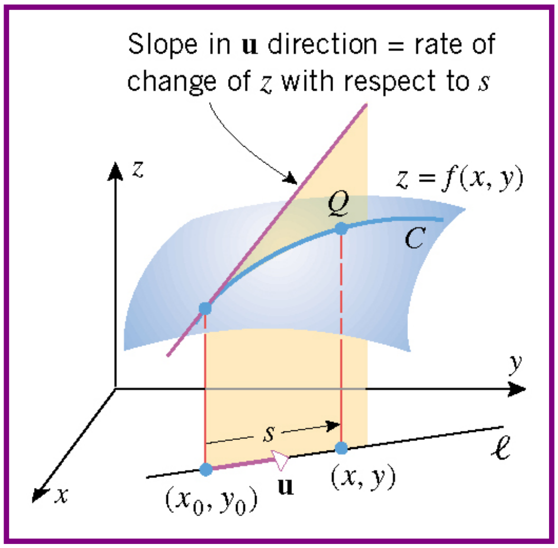
- rate of change of 'f' in any given direction

What is directional derivative

in the direction of vector $\vec{\mathbf{u}}$ at (x_0, y_0)

- Slope of surface $z = f(x, y)$ at (x_0, y_0) in the direction of $\vec{\mathbf{u}}$
- Rate of change of $z = f(x, y)$ at (x_0, y_0) in the direction of $\vec{\mathbf{u}}$

See class explanation



Main question:

How to compute directional derivatives?

How to compute directional derivative

The directional derivative of $f(x,y)$ in the direction of **unit vector** $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ is given by

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

Assumption

f(x,y) differentiable

The directional derivative of $f(x,y,z)$ in the direction of **unit vector** $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\mathbf{u}} f(x,y,z) = f_x(x,y,z)u_1 + f_y(x,y,z)u_2 + f_z(x,y,z)u_3$$

Similar formulas for more variables

Example 14.6.1 Find directional derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of $\theta = \frac{\pi}{3}$.

Solution Done in class

Example 14.6.2 Find directional derivative of $f(x,y,z) = x^3z + y^3z^2 - xyz$ in the direction of $\vec{\mathbf{v}} = \langle -1,0,3 \rangle$.

Note to use unit vector to find directional derivatives

Solution Done in class

Gradient vector of a function

Also called gradient of ‘f’

For a function $f(x,y,z)$, the gradient vector of $f(x,y,z)$ is defined as

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Similar definition for functions of two variables or more variables.

Directional derivative in terms of gradient

The directional derivative of $f(x,y,z)$ in the direction of **unit vector** $\vec{\mathbf{u}} = < u_1, u_2, u_3 >$ is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \vec{\mathbf{u}}$$

Obvious from last page

Example 14.6.3 If $f(x,y) = x\cos y$.

(a) Find the gradient of f.

(b) Find directional derivative of f at $(1,0)$ in the direction of $\vec{\mathbf{v}} = \langle 2,1 \rangle$.

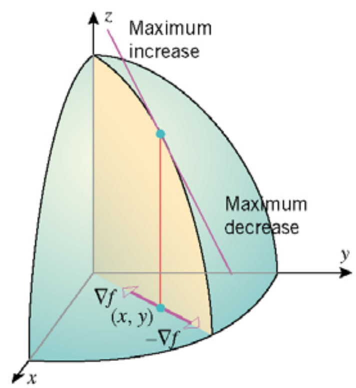
Solution Done in class

Important fact-1 about gradient

Gradient vector determines the maximum/minimum rate of change of a function

Let 'f' be a function of 2 0r 3 variables.

- The maximum value of the directional derivative of 'f' occurs in direction of gradient vector ∇f .
- Hence, the maximum value of the directional derivative of 'f' (i.e. maximum rate of change of 'f') is $|\nabla f|$.



Why?

Since $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$

Example 14.6.4 Find the direction in which $f(x, y, z) = x^3 z^2 + y^3 z + z - 1$ increases most rapidly at $P(1,1,-1)$. Find the rate of change at $P(1,1,-1)$ in that direction.

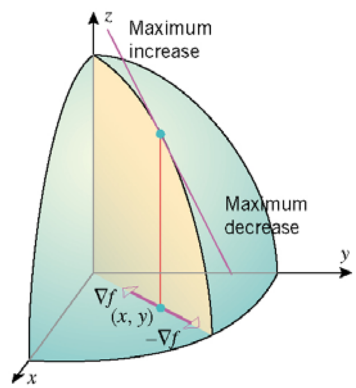
Solution Done in class

Important fact-1 about gradient (continued)

Gradient vector determines the maximum/minimum rate of change of a function

Let '*f*' be a function of 2 0r 3 variables.

- The minimum value of the directional derivative of '*f*' occurs in direction opposite to that of gradient vector ∇f .
- Hence, the minimum value of the directional derivative of '*f*' (i.e. minimum rate of change of '*f*') is '$-|\nabla f|$'.



Why?

Since $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$

Exercise Find the direction in which $f(x,y,z) = 4e^{xy}\cos z$ decreases most rapidly at $P(0,1,\frac{\pi}{4})$. Find the rate of change at $P(0,1,\frac{\pi}{4})$ in that direction.

Important fact-2 about gradient

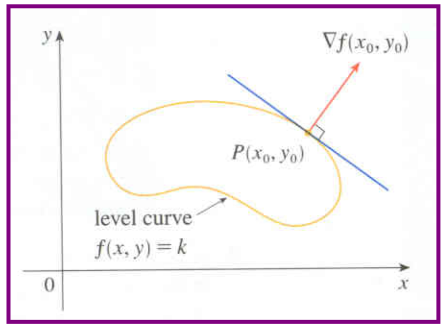
Given a surface $z = f(x,y)$.

- Of course the direction $\nabla f\left(x_0, y_0\right)$ is important
- How to determine this direction from the contour map of $f(x,y)$

Also see e.g. below

Recall that for $z = f(x,y)$ we can find the level curve $f(x,y) = k$ that passes through the point (x_0, y_0)

- Let $z = f(x,y)$ be a surface and $f(x,y) = k$ be the level curve that passes through (x_0, y_0).
- Then $\nabla f\left(x_0, y_0\right)$ is orthogonal to the level curve $f(x,y) = k$



Example 14.6.5

(a) Find and sketch the level curve of $f(x,y) = x^2 + 4y^2$ at $P(-2,0)$

(b) Draw the gradient vector at P.

Solution Done in class

Important fact-2 about gradient (continued)

- Let $w = f(x, y, z)$ be a function of three variables.
- Then $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level curve $f(x, y, z) = k$

  (which passes through (x_0, y_0, z_0))

This important fact will be used in the next section

Do Qs: 1-60

End of Section 14.6