Section 14.5 *Chain rule*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. apply chain rule for functions of two or more variables

Recall chain rule for functions of 1-variable

If $y = f(x)$ and $x = g(t)$ then $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

- Here we study chain rule for functions of more variables
- It has different versions

Chain rule **Case-I**

If $z = f(x,y)$ and $x = g(t)$, $y = h(t)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

see tree diagram explanation in class

If $w = f(x,y,z)$ and $x = g(t)$, $y = h(t)$, $z = k(t)$ then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

see tree diagram explanation in class

Example 14.5.1 Find $\frac{dz}{dt}$ if $z = xe^{xy}$ and $x = t^2$, $y = t^{-1}$.

Solution Done in class

Chain rule **Case-II**

If $z = f(x,y)$ and $x = g(u,v)$, $y = h(u,v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

see tree diagram explanation in class

If $w = f(x,y,z)$ and $x = g(u,v)$, $y = h(u,v)$, $z = k(u,v)$ then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

see tree diagram explanation in class

Example 14.5.2 Let $w = e^{xyz}$ and $x = 3u + v$, $y = 3u - v$, $z = u^2 v$. Find $\frac{\partial w}{\partial v}$.

Solution Done in class

Extension to other situations

Similar to above logic you should be able to develop the chain rule for different situations.

Exercise Let $w = f(x,y,z)$ and $x = g(t,u,v)$, $y = h(t,u,v)$, $z = k(t,u,v)$.

Write down the chain rule formula for $\frac{\partial w}{\partial t}$.

Answer: $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$

Implicit differentiation

- Given $F(x,y)=C$

Defining y implicitly as function of x

- Differentiating w.r.t. x we get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-F_x}{F_y}$$

If $f(x,y)=C$ implicitly defines y as a function of x then

$$\frac{dy}{dx}=\frac{-F_x}{F_y} \qquad (\text{if } F_y\neq 0)$$

Example 14.5.3 Find $\frac{dy}{dx}$ from $x\cos 3y+x^3y^5=3x-e^{xy}$

Solution Done in class

Defining z implicitly as function of x and y

- Consider $F(x,y,z)=C$ (*)
  - Differentiate (using chain rule) Eq. (*) w.r.t. x and show that

$$\frac{\partial z}{\partial x}=\frac{-F_x}{F_z}$$

See Q.35 in the book

  - Differentiate (using chain rule) Eq. (*) w.r.t. y and show that

$$\frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$$

See Q.36 in the book

Exercise Let $x^2\sin(2y-5z)=1+y\cos(6xz)$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

End of Section 14.5