Section 14.3 *Partial derivatives*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. understand the meaning of partial derivatives
2. calculate partial derivatives
3. know the physical interpretation of partial derivatives

Partial derivatives: Definition

Formal Definition

- The partial derivative of $f(x,y)$ w.r.t. 'x' at the point (x_0, y_0) is denoted as $\frac{\partial f}{\partial x}$ and is defined as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Rate of change in 'x' direction

- The partial derivative of $f(x,y)$ w.r.t. 'y' at the point (x_0, y_0) is denoted as $\frac{\partial f}{\partial y}$ and is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Rate of change in 'x' direction

Informal Definition $\frac{\partial f}{\partial x}$

Ordinary derivative of $f(x,y)$ w.r.t. 'x'

- keeping 'y' as constant

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Ordinary derivative of $f(x,y)$ w.r.t. 'y'

- keeping 'x' as constant

Partial derivatives: Basic computations

Notation

Consider $z = f(x,y)$.

- The partial derivatives w.r.t. 'x' and 'y' are denoted respectively by

$$f_x,\ \frac{\partial f}{\partial x},\ \frac{\partial z}{\partial x} \qquad \text{and} \qquad f_y,\ \frac{\partial f}{\partial y},\ \frac{\partial z}{\partial y}$$

- The partial derivatives at the point (x_0, y_0) are

$$f_x(x_0,y_0),\ \left.\frac{\partial f}{\partial x}\right|_{x=x_0,\ y=y_0},\ \left.\frac{\partial z}{\partial x}\right|_{(x_0,\,y_0)} \qquad \text{and} \qquad f_y(x_0,y_0),\ \left.\frac{\partial f}{\partial y}\right|_{x=x_0,\ y=y_0},\ \left.\frac{\partial z}{\partial y}\right|_{(x_0,\,y_0)}$$

See Examples 14.3.1, 14.3.2 done in class

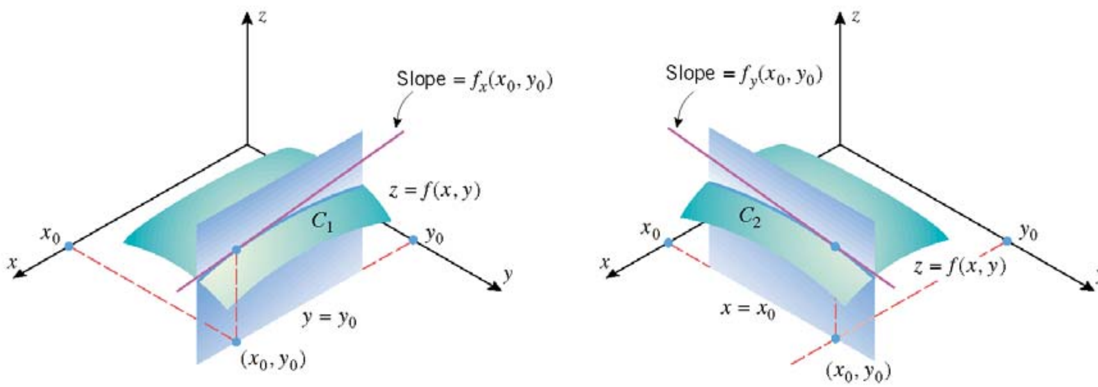
Implicit partial differentiation

- Learn through an example

Example 14.3.3 Consider $x^3z^2 - 5xy^5z = x^2 + y^3$. Find $\frac{\partial z}{\partial x}$.

Exercise Consider $x^2\sin(2y-5z) = 1 + y\cos(6xz)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Interpretation of partial derivatives of z = f(x,y)



$\frac{\partial f}{\partial x}(x_0, y_0)$

- **rate of change** of *f(x,y)* in **x-direction**
- **slope** of surface *z=f(x,y)* in **x-direction**

$\frac{\partial f}{\partial y}(x_0, y_0)$

- **rate of change** of *f(x,y)* in **y-direction**
- **slope** of surface *z=f(x,y)* in **y-direction**

Examples 14.3.4 Find the slope of the tangent line at $(-1,1,5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and

(a) the plane $x = -1$ — i.e. y-direction

(b) the plane $y = 1$ — i.e. x-direction

Solution Done

Higher order partial derivatives

- Since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are also functions of *x* & *y*, so we can differentiate them further
- For *z=f(x,y)*, the four second order partial derivatives are
  - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$
  - $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$
  - $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$
  - $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$

Called mixed partial derivatives

If the function '*f*' is nice then the order in mixed derivatives is not important, i.e. $f_{xy} = f_{yx}$

Equality of mixed Partial derivatives

Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

Examples 14.3.5 The 1st order partial derivative of $f(x,y) = \cos 2x - x^2 e^{5y} + 3y^2$ are $f_x = -2\sin 2x - 2xe^{5y}$ and $f_y = -5x^2 e^{5y} + 6y$. Find all 2nd order partial derivatives.

Solution Done in class

Exercise Let $f(x,y) = xe^{-x^2 y^2}$. Verify that $f_{xy} = f_{yx}$.

Partial derivatives of functions of more than two variables

- Until now we have only studied partial derivatives of functions of two variables.
- But the concept & computations of partial derivatives of functions of more than two variables are similar. [See example below]

Examples 14.3.6 Calculate f_x and f_{xz} for $f(x,y,z) = z^3 y^2 \ln x$.

Solution Done in class

Exercise Calculate f_{xxyzz} for $f(x,y,z) = z^3 y^2 \ln x$.

Answer: $\dfrac{-12yz}{x^3}$

Partial differential equations

Equations involving partial derivatives

Some important examples of partial differential equations are

- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ Laplace's equation
- $\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$ heat equation
- $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ Wave equation

Exercise Show that $z = x^2 - y^2 + 2xy$ satisfies Laplace's equation.

End of Section 14.3

Do Qs similar to HW problems