Section 12.6 *Planes in 3-space*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know point-normal form of equation of a plane
2. find equation of plane using different given information
3. determine whether or not given planes are parallel or perpendicular
4. find line of intersection of two planes
5. find distance
   a. of a point from a plane
   b. between two parallel planes
   c. between two skew lines

Equation of a plane

To determine a plane we require

- a point on the plane
- a vector normal to the plane

See class explanation

The equation of a plane passing through $P_0(x_0, y_0, z_0)$ and having the normal vector $\vec{v} = \langle a, b, c \rangle$ is

Point-normal form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (*)$$

Can write (*) as
$ax + by + cz = d$
where
$d = ax_0 + by_0 + cz_0$

general form

$$ax + by + cz = d \quad (**)$$

always gives a plane with normal $\vec{v} = \langle a, b, c \rangle$

Example 12.6.1 Find the equation of the plane through the point $P(5, -2, 4)$ and perpendicular to $\vec{n} = \langle 4, 2, 3 \rangle$.

Solution Done in class.

Example 12.6.2 Find the equation of the plane through the points $P_1(1, -2, 0)$, $P_2(3, 1, 4)$ and $P_3(0, -1, 2)$.

Solution Done in class.

Exercise 12.6.3 Find the equation of the plane that passes through $P_1(6, 0, -2)$ and contains the line $x = 4 - 2t,\ y = 3 + 5t,\ z = 7 + 4t$.

Solution Hint: Find two points P_2, P_3 on the line and use idea of previous example. Answer: $33x + 10y + 4z = 190$

Parallel or perpendicular planes

Two planes are

- parallel if their normals are parallel
- perpendicular if their normals are perpendicular

Example 12.6.4 Are the planes $x - y + 3z - 2 = 0$ and $2x + z = 1$

(a) parallel (b) perpendicular

Solution Done in class.

Example 12.6.5 Find equation of the plane through $P(5, -2, 4)$ that is parallel to the plane $3x + y - 6z + 8 = 0$

Solution Done in class.

Example 12.6.6 Find equation of the plane through the points $P_1(-2, 1, 4)$, $P_2(1, 0, 3)$ that is perpendicular to the plane $4x - y + 3z = 2$.

Solution Done in class.

Example 12.6.7 Find equation of the plane through $P(-1, 2, -5)$ that is perpendicular to the planes $2x - y + z = 1$ and $x + y - 2z = 3$.

Solution Done in class.

Line of intersection of two planes

Can you see how two planes intersect in a line

How to find line of intersection: understand through examples

Method 1: Solving system of two linear equations

Example 12.6.8 Find the equation of line of intersection of the planes

$$P_1: \quad x - y + 4z = 3$$
$$P_2: \quad 2x + y - z = -3$$

Solution Done in class.

Method 2: Using the following important fact

If P_1, P_2 are two intersecting planes with normals $\vec{n}_1, \vec{n}_2$ then

$\vec{n}_1 \times \vec{n}_2$ is parallel to the line of intersection of P_1, P_2.

Example 12.6.9 (Using method 2) Find the equation of line of intersection of the planes

$$P_1: \quad x - y + 4z = 3$$
$$P_2: \quad 2x + y - z = -3$$

Solution Done in class.

Exercise 12.6.10 Find the equation of the plane through $P(1, 2, -1)$ that is perpendicular to the line of intersection of the planes

$$P_1: \quad 2x + y + z = 2$$
$$P_2: \quad x + 2y + z = 3$$

Angle between two intersecting planes

The acute angle θ between planes P_1, P_2 is given by

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}$$

where $\vec{n}_1$, $\vec{n}_2$ are normals to planes P_1, P_2.

See class explanation

Example 12.6.11 Find the acute angle between the planes

P_1: $2x - 4y + 4z = 7$

P_2: $6x + 2y - 3z = 2$

Solution Done in class.

Distance between a point and a plane

The distance between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (**)$$

See class explanation

Example 12.6.12 Find the distance between the point $P_0(0, -1, 1)$ and the plane

$2x - 3y + z = 6$.

Solution Done in class.

Distance between two parallel planes

- Choose a point $P_0(x_0, y_0, z_0)$ on one plane
- Find the distance of $P_0(x_0, y_0, z_0)$ from the other plane [using Formula (**) on previous page]

Exercise 12.6.13 Show that the planes

$$P_1: \quad 4x - 6y + 2z = 8$$
$$P_2: \quad 2x - 3y + z = 6$$

are parallel. Find the distance between the two planes.

Answer: $\frac{2}{\sqrt{14}}$

Distance between two skew lines

- Two skew lines L_1, L_2 can be viewed as lying in two parallel planes P_1, P_2
- So the question of finding distance between L_1 and L_2 is equivalent to finding distance between parallel planes P_1 and P_2

What to do

- Find parallel planes P_1, P_2 containing skew lines L_1, L_2
- Find distance between parallel planes P_1, P_2 using above idea.

Example 12.6.14 Find the distance between skew lines

$$L_1: \quad x = 1 + t, \; y = -2 + 3t, \; z = 4 - t$$
$$L_2: \quad x = 2t, \quad y = 3 + t, \quad z = -3 + 4t$$

Solution Done in class.

End of Section 12.6

Do Qs: 1-47