Section 12.5 *Parametric equations of lines*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know equation of line in parametric form and symmetric form
2. find parametric equations of lines using different given information
3. learn how to find intersection of a line with other curves, planes and surfaces
4. learn how to find points of intersection of two lines
5. determine whether or not given line are parallel or perpendicular or intersecting or skew
6. find distance of a point from a given line

Parametric form of equation of a line

To determine a line we require

- a point on the line
- a vector parallel to the line

See class explanation

Parametric form

The equation of a line 'L' passing through $P_0(x_0, y_0, z_0)$ and parallel to vector $\vec{v} = \langle a, b, c \rangle$ is given (in parametric form) by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Similar definition in 2-space

Different values of parameter 't' give different points on the line.

Example 12.5.1 Find the parametric equation of the line 'L' that passes through the point $P_0(5,1,3)$ and is parallel to $\vec{v} = \langle 1, 4, -2 \rangle$. Find two points on the line.

Solution Done in class.

Example 12.5.2 Find a vector parallel to the line

$$x = 1 + 4t,\ y = 2 + 5t,\ z = -3 - 7t$$

Solution Done in class.

Exercise 12.5.3 Find the equation of line passing through origin and parallel to the line $x = 2t,\ y = 1 - t,\ z = 4 + 3t$.

Example 12.5.4 Find the equation of a line that is tangent to $y = x^2$ at $P_0(-2, 4)$.

Solution Done in class.

Symmetric form of equation of a line

Similar definition in 2-space

Symmetric form

The symmetric form of the equation of a line 'L' passing through $P_0(x_0, y_0, z_0)$ and parallel to vector $\vec{v} = \langle a, b, c \rangle$ is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

See class explanation

Example 12.5.5 Find the equation of a line through the points $P_1(1, 4, -3)$ and $P_2(2, -1, 3)$ in

a) symmetric form

b) parametric form

Solution Done in class.

- Since the two forms of equation of line are equivalent we will do our further study of lines only for parametric form.

Different types of lines

The lines 'L_1', 'L_2' are called

- parallel if their corresponding vectors $\vec{v}_1, \vec{v}_2$ are parallel i.e. multiple of each other
- perpendicular if their corresponding vectors $\vec{v}_1, \vec{v}_2$ are perpendicular i.e. $\vec{v}_1 \cdot \vec{v}_2 = 0$
- intersecting if these intersect at a point
- skew if these are neither parallel nor intersecting (therefore do not lie in same plane)

Example 12.5.6 Are the following lines parallel

a) $L_1: \; x = -2 + 3t, \; y = 1 - 6t, \; z = 4 + 5t$

$L_2: \; x = -5 - 6t, \; y = 3 + 12t, \; z = -7 - 10t$

b) $L_1: \; x = -2 - t, \; y = 1 + 2t, \; z = 2 + 2t$

$L_2: \; x = 1 + t, \; y = 2 - t, \; z = 1 + 3t$

Solution Done in class.

Example 12.5.7 Find the equation of line that passes through $P(0,2,1)$ and intersects the line $L: \; x = 2t, \; y = 1 - t, \; z = 2 + t$ at right angle.

Solution Done in class.

Intersection of a line with other curves, planes and surfaces

Learn the method through examples.

Example 12.5.8 Find the points where the line

$$L:\ x = 1 + t,\ y = 3 - t,\ z = 2t \qquad (1)$$

intersects the cylinder

$$x^2 + y^2 = 16 \qquad (*)$$

To solve, we need to answer the following question:

"Is there a value of t for which x, y, z satisfy the equation

$x^2 + y^2 = 16$ of the cylinder"

Solution Done in class.

Exercise 12.5.9 Find the points of intersection of

$$L:\ x = 1 + 3t,\ y = -1 + t,\ z = 2 - 2t$$

with the plane

$$2x - 3y + z = 6.$$

Answer: $(-2, -2, 4)$

Finding intersection of two line (given in parametric form)

Given two lines

$$L_1: \quad x = x_0 + a_0 t,\ y = y_0 + b_0 t,\ z = z_0 + c_0 t$$

$$L_2: \quad x = x_1 + a_1 s,\ y = y_1 + b_1 s,\ z = z_1 + c_1 s$$

Important to use different parameters

- L_1 and L_2 intersect if there exist values of s, t such that

$$x_0 + a_0 t = x_1 + a_1 s \qquad (1)$$
$$y_0 + b_0 t = y_1 + b_1 s \qquad (2)$$
$$z_0 + c_0 t = z_1 + c_1 s \qquad (3)$$

- We solve any two equations to get value of t, s
- Then we put these values in the 3rd equation:
  - if 3rd equation is satisfied then lines intersect
  - if 3rd equation is not satisfied then lines do not intersect

Example 12.5.10 Find the points of intersection of

$$L_1: x = 1 + 2t,\ y = 3 - 2t,\ z = -6 + 2t$$

$$L_2: x = -1 + 4s,\ y = -2 + 3s,\ z = -10 + 6s$$

Solution Done in class.

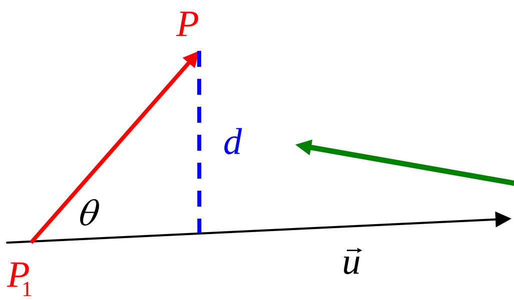
Example 12.5.11 Are the following lines skew?

$$L_1: x = 1 + t,\ y = -2 + 3t,\ z = 4 - t$$

$$L_2: x = 2s,\ y = 3 + s,\ z = -3 + 4s$$

Solution Done in class.

Finding distance between a point and a line in 3-space



Given a point P and a line L.

- Let P_1 be a point on L and $\vec{u}$ a vector along L
- If d is the distance between P and L then

$$d = \left\|\overrightarrow{P_1P}\right\|\sin\theta = \frac{\left\|\overrightarrow{P_1P}\times\vec{u}\right\|}{\|\vec{u}\|}$$

To find distance 'd' of a point P from a line L

- Choose a point P_1 on L
- Find a vector $\vec{u}$ along L
- Then $d = \dfrac{\left\|\overrightarrow{P_1P}\times\vec{u}\right\|}{\|\vec{u}\|}$

Example 12.5.12 Find the distance between $P(-2,1,1)$ and the line

$$L:\ x = 3 - t,\ y = t,\ z = 1 + 2t$$

Solution Done in class.

Finding distance between two parallel lines

- Take a point P on one line
- Find distance of P from other line by the method shown above.

End of Section 12.5

Do Qs. 1-48.