# Section 12.4 *Cross product*



## Learning outcomes

After completing this section, you will inshaAllah be able to

1. know meaning of vector product and its basic properties & facts
2. apply cross product to find areas of parallelograms and triangles
3. know what is scalar triple product
4. apply scalar triple product to find volume of parallelepipeds.

## Definition

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the

cross product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle$$

Result: a vector

**Easier way to remember**

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

**Example 12.4.1** Consider $\vec{u} = \langle 2, 1, -1 \rangle$, $\vec{v} = \langle -3, 4, 1 \rangle$

a. Find $\vec{u} \times \vec{v}$

b. Find $\vec{v} \times \vec{u}$

c. Find $(\vec{u} \times \vec{v}) \cdot \vec{u}$ and $(\vec{u} \times \vec{v}) \cdot \vec{v}$

**Solution** Done in class.

**Important fact 1**

$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

**Exercise 12.4.2** Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

## Basic properties of vector product

For any vector $\vec{u}$, $\vec{v}$ and $\vec{w}$ and any scalar $k$, the following relations hold:

- $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$
- $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$
- $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$
- $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$
- $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$
- $\vec{u}\times\vec{u} = \vec{0}$

## Geometric description of vector product

If $\theta$ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$$

assumption $0\le\theta\le\pi$

- Direction of $\vec{u}\times\vec{v}$ given by right hand rule
- See explanation of "right hand rule" in class

**Important fact 2**

$\vec{u}\times\vec{v} = 0 \quad \Leftrightarrow \quad \vec{u}$ and $\vec{v}$ are parallel

**Exercise 12.4.3** Check if the whether or not the vectors $\vec{u} = \langle -1,1,1\rangle$, $\vec{v} = \langle 1,2,3\rangle$ are parallel.

Answer: Not parallel

# Application of cross product

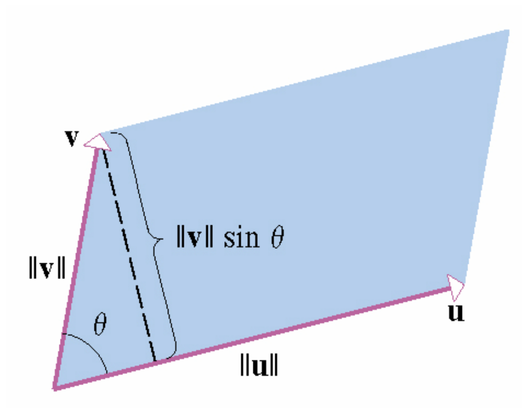
- **Finding area of a parallelogram**

Recall

- The area of a parallelogram determined by vectors $\vec{u}, \vec{v}$ is given by

$$A = \text{base} \times \text{altitude}$$

- If $\theta$ is the angle between $\vec{u}, \vec{v}$ then from figure
  - altitude $= \|\vec{v}\| \sin\theta$
  - base $= \|\vec{u}\|$
  - area $= \|\vec{u}\|\|\vec{v}\|\sin\theta = \|\vec{u} \times \vec{v}\|$




The **area of a parallelogram** determined by $\vec{u}$ and $\vec{v}$ is

$$A = \|\vec{u} \times \vec{v}\|$$

The **area of triangle** having adjacent sides given by $\vec{u}$ and $\vec{v}$ is

$$A = \frac{1}{2}\|\vec{u} \times \vec{v}\|$$

**Example 12.4.4** Find the area of triangle with vertices $P_1 = (1,4,6)$, $P_2 = (-2,5,-1)$, $P_3 = (1,-1,1)$.

**Solution** Done in class.

## Scalar triple product

The product $\vec{u}\cdot(\vec{v}\times\vec{w})$ is called

scalar triple product of $\vec{u}$, $\vec{v}$ *and* $\vec{w}$

Result: a scalar

**Efficient way of computing**

If $\vec{u}=\langle u_1,u_2,u_3\rangle$, $\vec{v}=\langle v_1,v_2,v_3\rangle$ and $\vec{w}=\langle w_1,w_2,w_3\rangle$ then

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

**Example 12.4.5** If $\vec{u}=\langle 1,2,3\rangle$, $\vec{v}=\langle 4,5,6\rangle$, $\vec{w}=\langle 7,8,0\rangle$, find $\vec{u}\cdot(\vec{v}\times\vec{w})$.

**Solution** Done in class.

**Important property**

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{w}\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\vec{w}\times\vec{u})$$

Why?

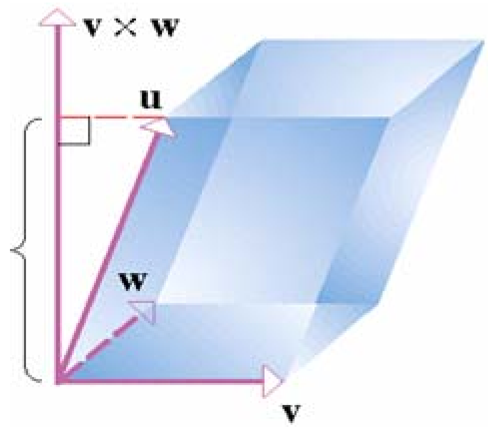
- one interchange of rows of determinant changes its value by '–1'
- here we make two interchanges

# Application of scalar triple product

- **Finding volume of a parallelepiped**

Recall

- The volume of a parallelepiped determined by vectors $\vec{u}, \vec{v}, \vec{w}$ is given by $V = \text{area of base} \times \text{altitude}$
- If $\theta$ is the angle between $\vec{u}$ and $\vec{v} \times \vec{w}$ then from figure
  - altitude $= \|\vec{u}\| |\cos\theta|$
  - area of base $= \|\vec{v} \times \vec{w}\|$
  - $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$




The **volume of a parallelepiped** determined by $\vec{u}, \vec{v}$ and $\vec{w}$ is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

**Important property**

$\vec{u}, \vec{v}$ and $\vec{w}$ are coplanar

$\Leftrightarrow \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

**Exercise 12.4.6** Find the volume of parallelepiped determined by $\vec{u} = \langle -5, 2, 4 \rangle$, $\vec{v} = \langle 1, 1, -2 \rangle$, $\vec{w} = \langle 2, -4, 6 \rangle$.

**Solution** Done in class.

**Exercise 12.4.7** Are the vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$, $\vec{w} = \langle 0, -9, 18 \rangle$ coplanar.

*End of Section 12.4*

*Do Qs. 1-30.*