Section 12.3 *Dot product: Projections*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know meaning of dot product and its geometric interpretation
2. apply dot product
   a. to find angle between two vectors
   b. to find direction cosines of a vector
   c. to find projection of a vector on another vector

Definitions and basic properties

- **Dot product**

Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Their dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad (*)$$

Similar definition for vectors in 2-space

Result is a scalar

Exercise 12.3.1 Find the dot product of

a. $\vec{u} = \langle 5, -8, 0 \rangle$, $\vec{v} = \langle 1, 2, 0 \rangle$ Answer: -11

b. $\vec{u} = \langle 0, 3, -7 \rangle$, $\vec{v} = \langle 2, 3, 1 \rangle$ Answer: 2

- **Properties of dot product**

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
5. $\vec{0} \cdot \vec{v} = 0$

- **Geometric interpretation in terms of angle between vectors**

If θ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta \qquad (**)$$

assumption $0 \le \theta \le \pi$

- Proof: Using law of cosines
- See book or Math002 notes

$\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow$ $\vec{u} \cdot \vec{v} = 0$

Exercise 12.3.2 Are the following vectors orthogonal?

a. $\vec{u} = \langle 2, 2, -1 \rangle$, $\vec{v} = \langle 5, -4, 2 \rangle$ Answer: Yes

b. $\vec{u} = \langle 3, -4, -1 \rangle$, $\vec{v} = \langle 0, 5, 2 \rangle$ Answer: No

Applications of dot product

- **Finding angle between two vectors**

From Eq. (**)

If θ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \qquad (***)$$

Example 12.3.3 Find the angle between $\vec{u} = \langle 2,2,-1\rangle$, $\vec{v} = \langle 5,-3,2\rangle$.

Solution Done in class.

why?

Observation

- $\vec{u}\cdot\vec{v} > 0 \Rightarrow$ angle between $\vec{u}$ and $\vec{v}$ is acute
- $\vec{u}\cdot\vec{v} < 0 \Rightarrow$ angle between $\vec{u}$ and $\vec{v}$ is obtuse
- $\vec{u}\cdot\vec{v} = 0 \Rightarrow$ $\vec{u}$ and $\vec{v}$ are orthogonal

Applications of dot product

- **Finding direction angles and direction cosines of a vector**

What are direction angles & direction cosines

Given a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$

- Direction angles α, β, γ are the angles which $\vec{v}$ makes with X-axis, Y-axis, Z-axis.
- The cosines $\cos\alpha, \cos\beta, \cos\gamma$ of direction angles are called direction cosines of $\vec{v}$.

From **(***),**

$$\cos\alpha = \frac{\vec{v}\cdot\mathbf{i}}{\|\vec{v}\|\|\mathbf{i}\|} = \frac{v_1}{\|\vec{v}\|}$$

$$\cos\beta = \frac{\vec{v}\cdot\mathbf{j}}{\|\vec{v}\|\|\mathbf{j}\|} = \frac{v_2}{\|\vec{v}\|}$$

$$\cos\gamma = \frac{\vec{v}\cdot\mathbf{k}}{\|\vec{v}\|\|\mathbf{k}\|} = \frac{v_3}{\|\vec{v}\|}$$

The **direction cosines** of $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|},\ \cos\beta = \frac{v_2}{\|\vec{v}\|},\ \cos\gamma = \frac{v_3}{\|\vec{v}\|}$$

Example 12.3.4 Find the direction cosines of $\vec{v} = \langle 2, 1, -4 \rangle$.

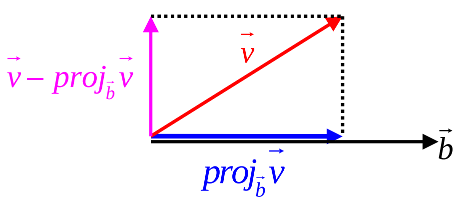
Solution Done in class.

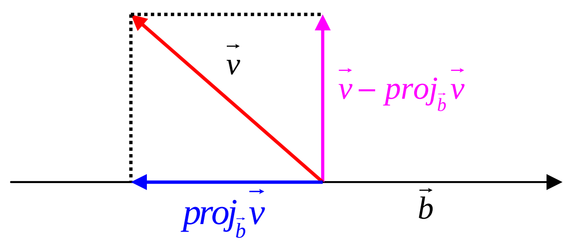
Applications of dot product (contd)

- **Orthogonal projection of a vector on another vector**

Q. What is orthogonal projection of $\vec{v}$ on $\vec{b}$ (geometrically)?

A. Roughly speaking shadow of $\vec{v}$ on $\vec{b}$ (see figures below & explanation in class)

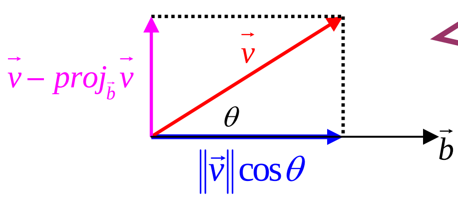




- **Using dot product to find projection of a vector on another**

Let θ be angle between $\vec{v}$ and $\vec{b}$.

Then



- Since $\|\vec{v}\|\cos\theta = \dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}$
- Hence $proj_{\vec{b}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|}\right)\left(\dfrac{\vec{b}}{\|\vec{b}\|}\right)$

The projection of $\vec{v}$ on $\vec{b}$ is given by $$proj_{\vec{b}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

Example 12.3.5 Find the projection of $\vec{v} = \langle 2,1,-1\rangle$ on $\vec{b} = \langle 1,0,-2\rangle$.

Solution Done in class.

Exercise What do you think should be the answer of the following dot product?

$$\left(\vec{v} - proj_{\vec{b}}\vec{v}\right) \cdot proj_{\vec{b}}\vec{v}$$

End of Section 12.3

Do Qs. 1-26, 35-42.