Section 12.2 *Vectors*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know definition and different representations of vectors
2. learn to perform basic vector operations in geometric as well as component form
3. understand what is meant by a unit vector and how to normalize a vector to get a unit vector
4. find vectors using information about their length and directions

- Most of the material in this section was covered in Math002 (Section 7.3).
- We will quickly refresh the material you already know (from Math002) and cover this section without going into much detail.

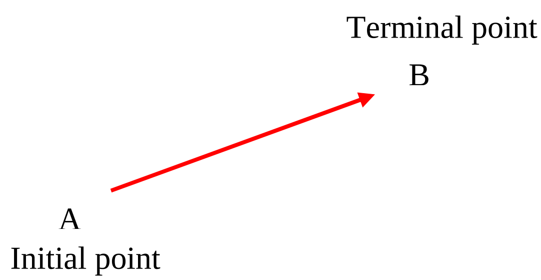
Basic definitions and ideas

- **Vectors**

  Describe quantities that have both magnitude and direction. e.g. Force, Velocity

- **Geometric representation of vectors**

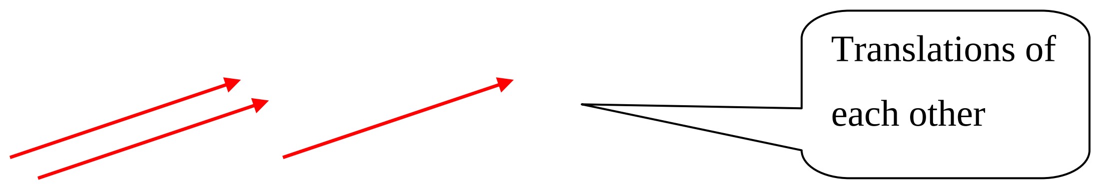
  By arrows (as shown in figure)



  - length of arrow = magnitude of vector
  - direction of arrow = direction of vector

- **Equivalent vectors**

  Having same magnitude and same direction



- **Zero vector $\vec{0}$**

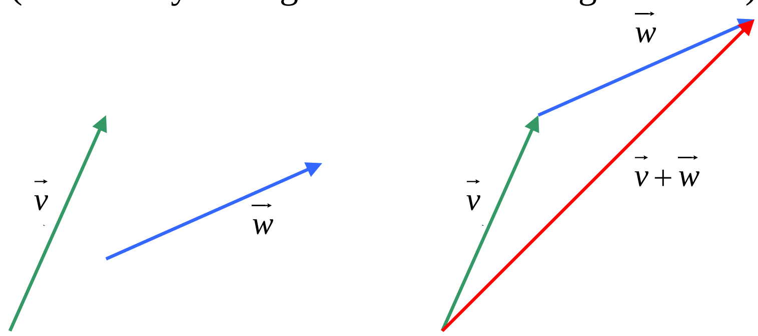
  Vector of length zero.

  Has no specific direction

Basic operations on vectors

- **Vector addition**

  (Defined by Triangle Law shown in figure below)



See explanation in class

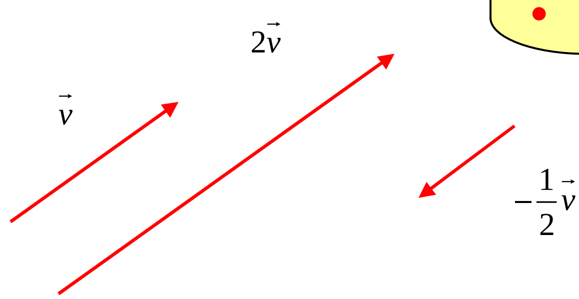
- **Scalar multiplication**

  Given a vector $\vec{v}$ and a scalar k. Then

  $k\vec{v}=$ a vector with

  - length $|k|$ times the length of $\vec{v}$
  - direction
    - same as $\vec{v}$ if $k>0$
    - opposite to $\vec{v}$ if $k<0$

Example:



- **Negative of a vector**

  $-\vec{v}=(-1)\vec{v}$ is called negative of $\vec{v}$

  Have same magnitude but opposite direction

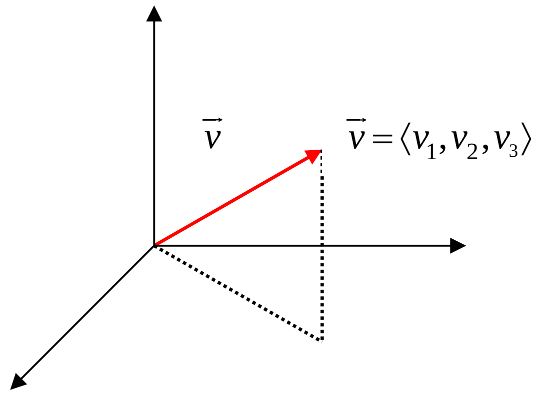
- **Difference of two vectors**

  $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$

Vectors in components

- **Components of a vector with initial point $O(0,0,0)$ and terminal point $P(v_1,v_2,v_3)$**

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$



- **Components of a vector with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$**

$$\vec{v} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- **Components of zero vector**

$$\langle 0,0,0 \rangle$$

- **Equivalent vectors**

Corresponding components are same

Exercise 12.2.1 Given $P_1(2,-7,0), P_2(1,-3,-5)$. Find the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$. Are the two vectors equivalent?

Arithmetic operations on vectors (in components)

If $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then

- $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- $k\vec{v} = \langle kv_1, kv_2, kv_3 \rangle$

Rules of vector arithmetic

For any vector $\vec{u}$, $\vec{v}$ and $\vec{w}$ and any scalars k and l , the following relations hold:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{0} + \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $k(l\vec{u}) = (kl)\vec{u}$
- $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
- $(k + l)\vec{u} = k\vec{u} + l\vec{u}$

Magnitude or length of a vector (in components)

Given $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the magnitude or length or norm of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Note: $\|c\vec{v}\| = |c|\|\vec{v}\|$.

Why?

Unit vector

A vector whose length is 1.

Normalizing a vector

Finding a unit vector in the direction of a given vector is called normalization.

How to normalize a vector

Given $\vec{v}$.

Then $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the direction of $\vec{v}$.

Example 12.2.2 Find a unit vector in the direction of $\vec{v} = \langle -5, 2, 1 \rangle$.

Solution: Done in class

Special unit vectors

$\mathbf{i} = \langle 1,0,0 \rangle$: unit vector along X-axis
$\mathbf{j} = \langle 0,1,0 \rangle$: unit vector along Y-axis
$\mathbf{k} = \langle 0,0,1 \rangle$: unit vector along Z-axis

Any vector in 3-space can be expressed in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$

Example 12.2.3

We can write $\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

because

$$\begin{aligned} \langle v_1, v_2, v_3 \rangle &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1,0,0 \rangle + v_2 \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle \end{aligned}$$

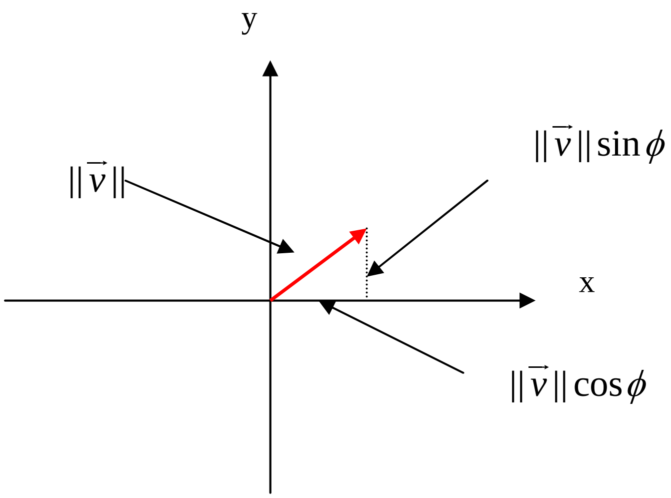
Exercise 12.2.4

If $\vec{v}_1 = \langle 1, 2, -3 \rangle$, $\vec{v}_2 = \langle 4, 0, 7 \rangle$, express $2\vec{v}_1 + 3\vec{v}_2$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Answer: $14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}$

- From 12.2_4 to 12.2_7, we have done the concepts and methods for vectors in 3-space.
- All this holds for vectors in 2-space in a similar fashion.

Finding a vector in 2-space if its length and angle with X-axis are known



- Given a vector $\vec{v}$ which makes angle ϕ with positive X-axis.
- Then $\vec{v}$ is given in component form as

$$\vec{v} = \langle \|\vec{v}\|\cos\phi, \|\vec{v}\|\sin\phi \rangle$$

Example 12.2.5 Find a vector $\vec{v}$ that makes angle $\frac{\pi}{3}$ with the X-axis and has magnitude $\|\vec{v}\| = 4$.

Solution:

$$\vec{v} = \left\langle \|\vec{v}\|\cos\frac{\pi}{3}, \|\vec{v}\|\sin\frac{\pi}{3} \right\rangle$$

$$\Rightarrow \quad \vec{v} = \left\langle 4\cdot\frac{1}{2}, 4\cdot\frac{\sqrt{3}}{2} \right\rangle = \langle 2, 2\sqrt{3} \rangle.$$

Finding a vector if its length and direction are known

If $\vec{u}$ is a unit vector in direction of $\vec{v}$ then

$$\vec{v} = \|\vec{v}\|\vec{u}$$

Example 12.2.6 Find a vector $\vec{v}$ of length $\sqrt{5}$ in the direction of $\vec{w} = 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

Solution: Done in class.

End of Section 12.2

Do Qs. 1-40