Section 11.2 *Tangent lines and arc lengths for parametric & polar curves*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. learn how to find slopes and tangent lines of parametric curves
2. learn how to find slopes and tangent lines of polar curves
3. know special trick to find tangent line to polar curves at pole
4. find arc length of polar curves

Tangent lines to parametric curves

Given a parametric curve $x = f(t), \;\; y = g(t); \qquad a \le t \le b$

- $\frac{dy}{dx}$ is defined when $\frac{dx}{dt} \neq 0$ and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- Similarly

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

* slope of tangent line at $x = f(t_0), \; y = g(t_0)$

$$\left.\frac{dy}{dx}\right|_{t=t_0}$$

* Tangent horizontal when

$$\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0$$

* Tangent vertical when

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} \neq 0$$

* Points where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ are called singular points

Example 11.2.1 Consider the parametric equation $x = t^2, \; y = t^3 - 3t$. Find

$\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $t = 1$.

Solution Done in class

Tangent lines to polar curves

Every polar equation $r = f(\theta)$ can be regarded as a parametric equation (with parameter θ)

$$x = r\cos\theta \quad \left(\text{or } x = f(\theta)\cos\theta\right)$$

$$y = r\sin\theta \quad \left(\text{or } x = f(\theta)\sin\theta\right)$$

- Then as above

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

* Tangent horizontal when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$
* Tangent vertical when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$
* Points where $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$ are called singular points

Example 11.2.2 Consider $r = 1 + \sin\theta$, $\quad 0 \le \theta \le 2\pi$.

a) Find the equation of tangent line at $\theta = 0$.

b) Find the points on the cardioid where tangent line is horizontal or vertical.

c) Find singular points.

Solution

- $y = r\sin\theta = (1+\sin\theta)\sin\theta$

$$\Rightarrow \quad \frac{dy}{d\theta} = (1+\sin\theta)\cos\theta + \cos\theta\sin\theta = \cos\theta(1+2\sin\theta)$$

- $x = r\cos\theta = (1+\sin\theta)\cos\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = (1+\sin\theta)(-\sin\theta) + \cos\theta\cos\theta$$

$$= -\sin\theta - \sin^2\theta + \cos^2\theta = -\sin\theta - \sin^2\theta + 1 - \sin^2\theta$$

$$= -2\sin^2\theta - \sin\theta + 1 = (1+\sin\theta)(1-2\sin\theta)$$

Rest of solution done in class

Tangent lines to polar curves at pole

- We have seen that slope of tangent line to $r = f(\theta)$ at $\theta = \theta_0$ is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \left. \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} \right|_{\theta=\theta_0} \qquad \textbf{(*)}$$

- We want to know equation of tangent at pole.
- Note that at pole we have $r = 0$.
- Suppose $r = 0$, when $\theta = \theta_0$. Then from (*), the slope of tangent line at pole (i.e. at the point $(0, \theta_0)$) is

$$\frac{dy}{dx} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$$

> But this is slope of the line $\theta = \theta_0$
>
> Hence the tangent line at origin is $\theta = \theta_0$

Conclusion

If θ_0 is a value of θ for which $r = f(\theta) = 0$

then

$\theta = \theta_0$ is tangent to $r = f(\theta)$ at origin.

To find all the tangents to $r = f(\theta)$ at origin, we need to **find all values of θ for which $f(\theta) = 0$.**

Example 11.2.3 Find tangent lines to $r = 2\sin 3\theta$ at pole for $0 \le \theta < \pi$.

Solution Done in class.

Arc length in polar

Recall from Math 102

For parametric curve $x = x(t),\ y = y(t)$, the arc length of the curve for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (*)$$

- (as on Slide 3) Regarding $r = f(\theta)$ as the parametric curve with parameter θ, we have

$$\begin{aligned} x &= r\cos\theta \quad (\text{or } x = f(\theta)\cos\theta) \\ y &= r\sin\theta \quad (\text{or } x = f(\theta)\sin\theta) \end{aligned}.$$

  This implies

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad (1)$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad (2)$$

- Using (1), (2) in (*) we get

Given a polar curve $r = f(\theta)$.

The arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

We assume that curve is traced only once

Example 11.2.4 Find the length of the cardioid $r = 1 + \sin\theta$.

Solution Done in class.

Exercise 11.2.5 Find the length of the $r = 2 - 2\cos\theta$.

Exercise 11.2.6 Find the length of the $r = 3\sin\theta$ for $0 \le \theta < \frac{\pi}{3}$.

Answer: π

End of 11.2

Do Qs: 1-46, 49-53