Section 11.1 *Polar coordinates*

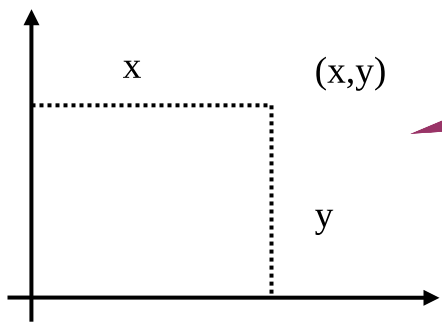
Learning outcomes

After completing this section, you will inshaAllah be able to

1. know what are polar coordinates
2. see the relation between rectangular and polar coordinates
3. learn how to graph polar curves using
   a. Method I: (from table of values)
   b. Method II: (by considering r, θ as rectangular coordinates)
   c. Method III: (by making use of symmetries in above two methods)
   d. MATLAB
4. know important families of polar curves

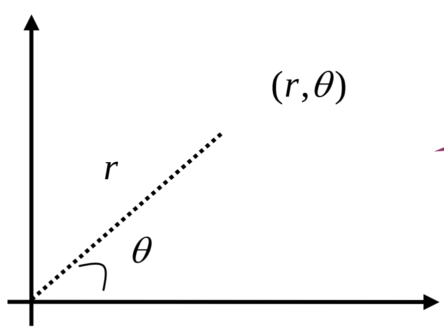
What are polar coordinates?

- Coordinate system: just a way to define a point.
- In rectangular coordinates a point 'P' is given by coordinates (x,y) which means



- start at origin
- move 'x' units horizontally and 'y' units vertically and we reach point 'P'

- Another way to define the point 'P' is as (r,θ) which means

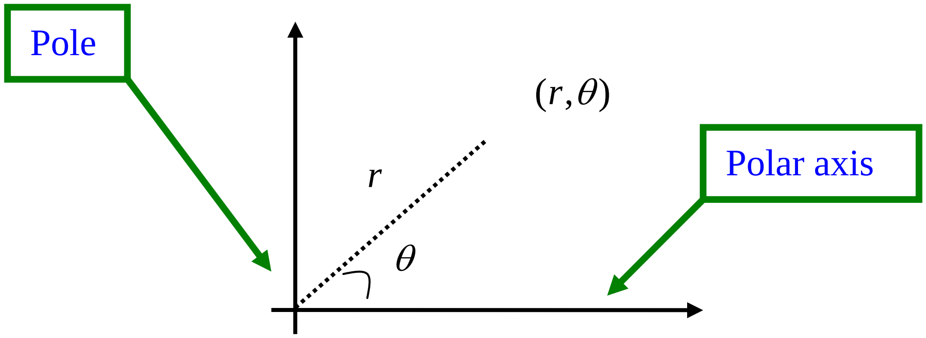


- start at origin
- move 'r' units along the line making angle θ with positive X-axis

Such coordinates are called polar coordinates of point 'P'

Standard terminology for polar coordinates

Given the polar coordinates (r,θ) of a point 'P'



- θ: Polar angle
  - positive anticlockwise
  - negative clockwise
- If $r>0$ then the point (r,θ) lies in the same quadrant as θ.

  If $r<0$ then the point (r,θ) lies in the quadrant opposite to that of θ.

Example 11.1.1 Plot the points $\left(3,\frac{\pi}{6}\right),\left(-3,\frac{\pi}{6}\right),\left(2,-\frac{2\pi}{3}\right),\left(-2,-\frac{2\pi}{3}\right)$.

Solution Done in class

> Important remark
>
> In rectangular coordinate system, each point has unique coordinates but in polar coordinate system a point has infinitely many coordinates

Example The coordinates (r,θ), $(r,\theta+2n\pi)$ and $\left(-r,\theta+2(n+1)\pi\right)$ all represent the same point. {can you see why?}

Example 11.1.2 Plot the points $\left(2,\frac{\pi}{4}\right),\left(2,\frac{9\pi}{4}\right),\left(2,\frac{17\pi}{4}\right)$.

Solution Done in class

Exercise 11.1.3 Do all the coordinates $\left(5,\frac{\pi}{3}\right),\left(5,-\frac{5\pi}{3}\right),\left(-5,\frac{4\pi}{3}\right),\left(-5,-\frac{2\pi}{3}\right)$ represent the same point?

Relation between rectangular & polar coordinates

- See explanation in the class

To find rectangular coordinates from polar coordinates we use

$$x = r\cos\theta$$
$$y = r\sin\theta \qquad (*)$$

Example 11.1.4 Find rectangular coordinates of $\left(-4, \frac{2\pi}{3}\right)$.

Solution Done in class

To find polar coordinates from rectangular coordinates we use

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\frac{y}{x} \qquad (**)$$

Follow from Eq. (*)

Important point to note

- $\tan^{-1}\frac{y}{x}$ only gives values of $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- But another angle $\theta + \pi$ is also possible
- See example below to understand this point

Example 11.1.5 Find polar coordinates of $(1,1)$ and $(-1,-1)$.

Solution Done in class

Relation between rectangular & polar coordinates (contd.)

The above formulas (*) and (**) can also be used to convert equations from one coordinate system to another.

Example 11.1.6 Express the following into rectangular coordinates.

1) $r = 3$

2) $r\sin\theta = 2$

3) $r = 3\cos\theta$

4) $r = \dfrac{6}{3\cos\theta + 2\sin\theta}$

Solution Done in class

Example 11.1.7 Convert $x^2 + y^2 - 6y = 0$ into polar coordinate system.

Solution Done in class

Graphing polar curves (Method I)

Method I to make graph of $r = f(\theta)$

- make a table for values of r, θ
- plot the points (r, θ) and join

Example 11.1.8 Sketch the curve $r = 2\cos\theta$.

Solution

* Table

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2

Note:

we have only used values of θ up to π because

if we take values of $\theta > \pi$ we get the same

.

* See graph in class

Exercise 11.1.9 Sketch the curve $r = 4\sin\theta$.

Graphing polar curves (Method II)

Method II to make graph of $r = f(\theta)$

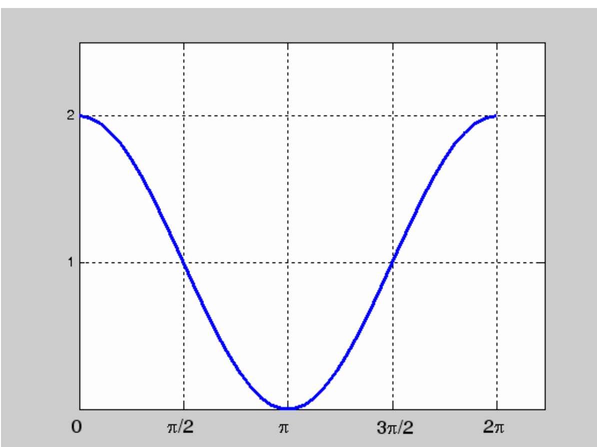
Step I First make the graph of $r = f(\theta)$ in rectangular θ, r- coordinate system (by considering (r, θ) as rectangular coordinates)

Step II Use the information from Step I to sketch the polar graph of $r = f(\theta)$

Example 11.1.10 Sketch the curve $r = 1 + \cos\theta$ for $0 \le \theta \le 2\pi$.

Solution

Step I Graph of $r = 1 + \cos\theta$ in rectangular coordinates



Useful information

- $r \ge 0$ for all values of θ
- $r = 2$ at $\theta = 0$

1. As θ increases from $0 \to \frac{\pi}{2}$, r decreases from $2 \to 1$
2. As θ increases from $\frac{\pi}{2} \to \pi$, r decreases from $1 \to 0$
3. As θ increases from $\pi \to \frac{3\pi}{2}$, r increases from $0 \to 1$
4. As θ increases from $\frac{3\pi}{2} \to 2\pi$, r increases from $1 \to 2$

Step II Draw polar graph of $r = 1 + \cos\theta$, using above information.

See class notes for the graph

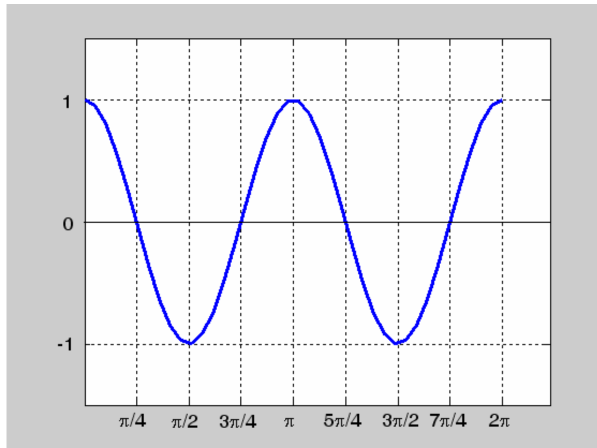
Graphing polar curves (Method II Contd.)

Example 11.1.11 Sketch the curve $r = \cos 2\theta$ for $0 \le \theta \le 2\pi$.

Solution

Step I

Graph of $r = \cos 2\theta$ in rectangular coordinates



Useful information

- $r = 1$ at $\theta = 0$

1. θ changes from $0 \to \frac{\pi}{4}$, r changes from $1 \to 0$
2. θ changes from $\frac{\pi}{4} \to \frac{\pi}{2}$, r changes from $0 \to -1$
3. θ changes from $\frac{\pi}{2} \to \frac{3\pi}{4}$, r changes from $-1 \to 0$
4. θ changes from $\frac{3\pi}{4} \to \pi$, r changes from $0 \to 1$
5. θ changes from $\pi \to \frac{5\pi}{4}$, r changes from $1 \to 0$
6. θ changes from $\frac{5\pi}{4} \to \frac{3\pi}{2}$, r changes from $0 \to -1$
7. θ changes from $\frac{3\pi}{2} \to \frac{7\pi}{4}$, r changes from $-1 \to 0$
8. θ changes from $\frac{7\pi}{4} \to 2\pi$, r changes from $0 \to 1$

Means distance from origin increases from 0 to 1 but this portion of curve lies in the quadrant opposite to that of θ.

Means distance from origin decreases from 1 to 0 but this portion of curve lies in the quadrant opposite to that of θ.

Step II Draw polar graph of $r = \cos 2\theta$, using above information.

See class notes for the graph

Symmetries of polar curves

See explanation in the class to understand the following ideas.

A polar curve $r = f(\theta)$ is symmetric

- about polar axis (or X-axis) if
  changing $\theta \to -\theta$
  $\Rightarrow$ no change in equation

  e.g. $r = \cos\theta$

- about vertical line $\theta = \frac{\pi}{2}$ (or Y-axis) if
  changing $\theta \to \pi - \theta$
  $\Rightarrow$ no change in equation

  e.g. $r = \sin\theta$

- about pole (or origin) if
  changing $r \to -r$ OR $\theta \to \pi + \theta$
  $\Rightarrow$ no change in equation

  e.g. $r = \sin 2\theta$, $r^2 = \cos\theta$

Symmetry about pole means curve remains unchanged if we rotate it through 180°

Graphing polar curves (Method III)

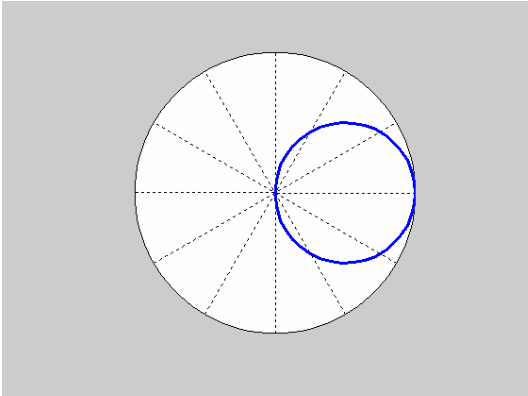
Method III

To use symmetry in Method I and Method II

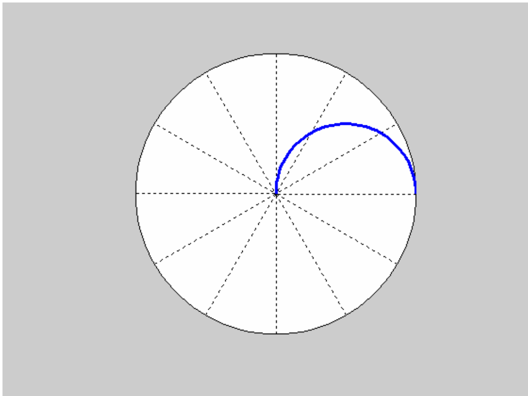
Example 11.1.12 The graph of $r = 2\cos\theta$ was sketched in Example 11.1.8 by using the following table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2

and the graph was



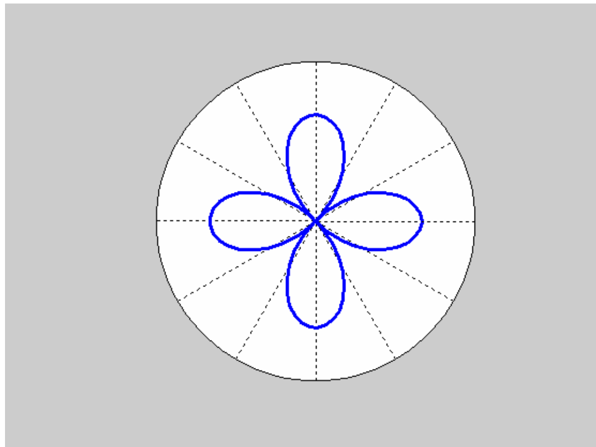
- Note: $r = 2\cos\theta$ is symmetric about polar axis (Why?) and for the values of θ from 0 to $\frac{\pi}{2}$ we get



- So we can complete the graph by using symmetry about polar axis and only using the values of θ from 0 to $\frac{\pi}{2}$ instead of using θ from 0 to π.

Graphing polar curves (Method III Contd.)

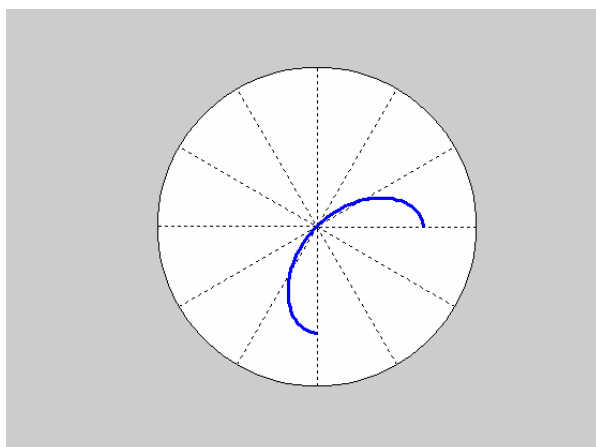
Example 11.1.13 The graph of $r = \cos 2\theta$ was sketched in Example 11.1.11 by using θ from 0 to 2π and the graph was



- Note: $r = \cos 2\theta$ is symmetric about pole as well as about polar axis and the line $\theta = \frac{\pi}{2}$ (why?) and for the values of θ from 0 to $\frac{\pi}{2}$ we get

- change $\theta \to \theta + \pi$
- change $\theta \to -\theta$
- change $\theta \to \pi - \theta$

and check.



- So we can complete the graph by using symmetries and above part of the curve.

Graphing polar curves (Important Exercise)

Exercise 11.1.14 Consider $r^2 = \cos 2\theta$ for $0 \le \theta \le 2\pi$.

This consists of two functions

$$r = \sqrt{\cos 2\theta} \qquad \text{and} \qquad r = -\sqrt{\cos 2\theta}.$$

a. Find symmetries of $r = \sqrt{\cos 2\theta}$.

Sketch $r = \sqrt{\cos 2\theta}$ by using symmetries in

- Method I
- Method II

b. Find symmetries of $r = -\sqrt{\cos 2\theta}$.

Sketch $r = \sqrt{\cos 2\theta}$ by using symmetries in

- Method I
- Method II

c. Do you get same graphs in Part (a) and (b)

d. What is the graph of $r^2 = \cos 2\theta$.

Before moving on to the next slide

- See Sections 1 and 2 of the "Introductory notes for Matlab beginners".
- See the handout "Plotting graphs in rectangular coordinates using Matlab"
- See the handout "Plotting polar curves using Matlab"

Available in **"Matlab Help"** area of WebCT

Graphing polar curves $r = f(\theta)$ (using MATLAB)

To know when does the curve or r begins to repeat itself

Step I Find the domain of θ

Find the smallest value of n so that

$$f(\theta + 2n\pi) = f(\theta)$$

and take

$$0 \le \theta \le 2n\pi$$

Step II Use MATLAB to make the polar graph

As explained in the handout "Plotting polar curves using Matlab"

Available in **"Matlab Help"** area of WebCT

Example 11.1.15 Use Matlab to plot $r = 2\sin 4\theta$.

Solution

Step 1 Domain of θ

We look at $2\sin\left(4(\theta + 2n\pi)\right) = 2\sin 4\theta$

$\Rightarrow \quad \sin\left(4\theta + 8n\pi\right) = \sin 4\theta$

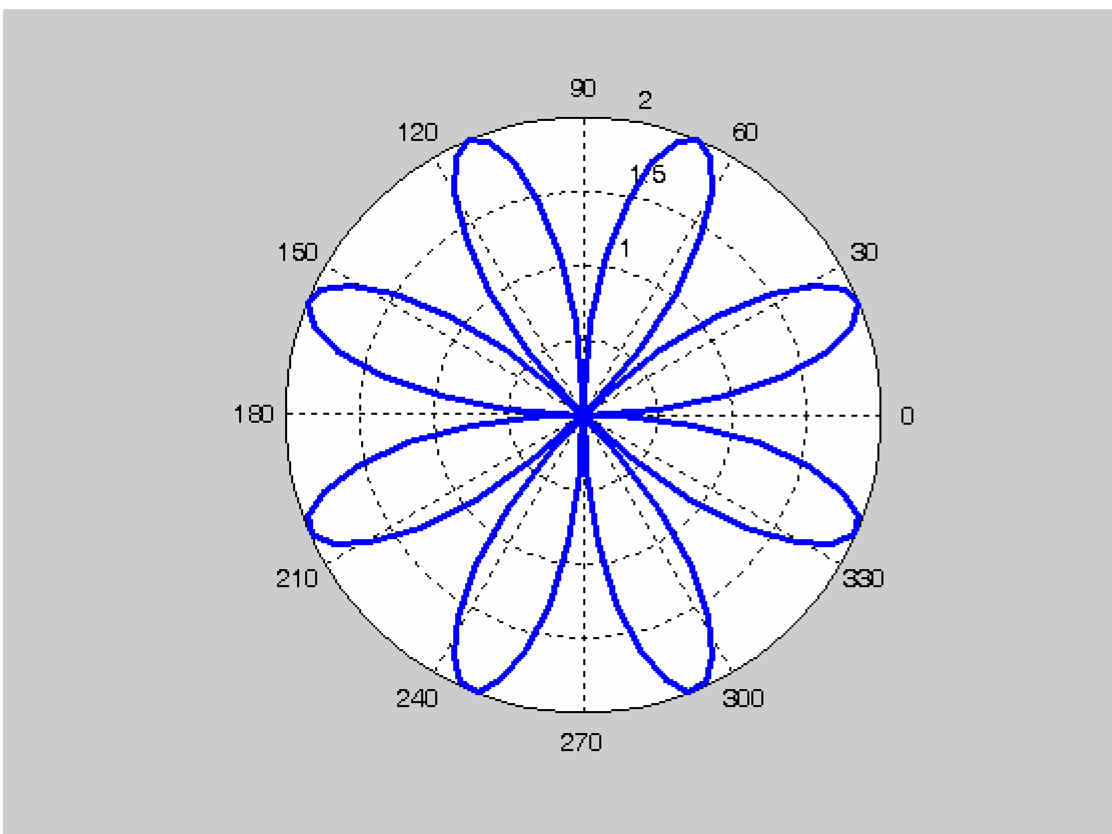
The smallest value of n for which $8n\pi$ is a multiple of 2π is $n = 1$.

so $0 \le \theta \le 2\pi$

Step 2

Use the following commands to get the graph

```
>>   theta=linspace(0,2*pi,100);
>>   r=2*sin(4*theta);
>>   polar(theta,r)
```



Example 11.1.16 Use Matlab to plot $r = 2\sin 5\theta$.

Solution

Step 1 Domain of θ

We look at $2\sin\left(5(\theta + 2n\pi)\right) = 2\sin 5\theta$

$$\Rightarrow \quad \sin\left(4\theta + 10n\pi\right) = \sin 4\theta$$

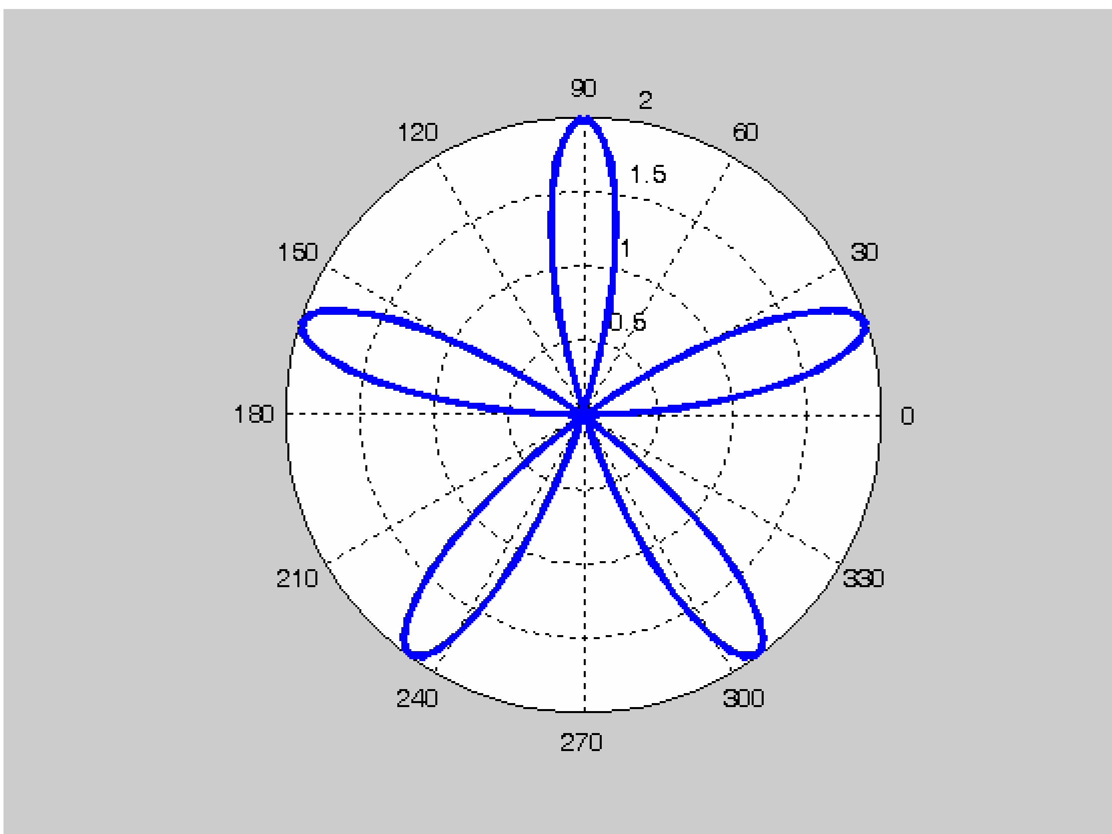
The smallest value of n for which $10n\pi$ is a multiple of 2π is $n = 1$.

so $0 \le \theta \le 2\pi$

Step 2

Use the following commands to get the graph

```
>>    theta=linspace(0,2*pi,100);
>>    r=2*sin(5*theta);
>>    polar(theta,r)
```



Example 11.1.17 Use Matlab to plot $r = \sin\left(\frac{8\theta}{5}\right)$.

Solution

Step 1 Domain of θ

We look at $\sin\left(\frac{8(\theta+2n\pi)}{5}\right) = \sin\left(\frac{8\theta}{5}\right)$

$\Rightarrow \quad \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\left(\frac{8\theta}{5}\right)$

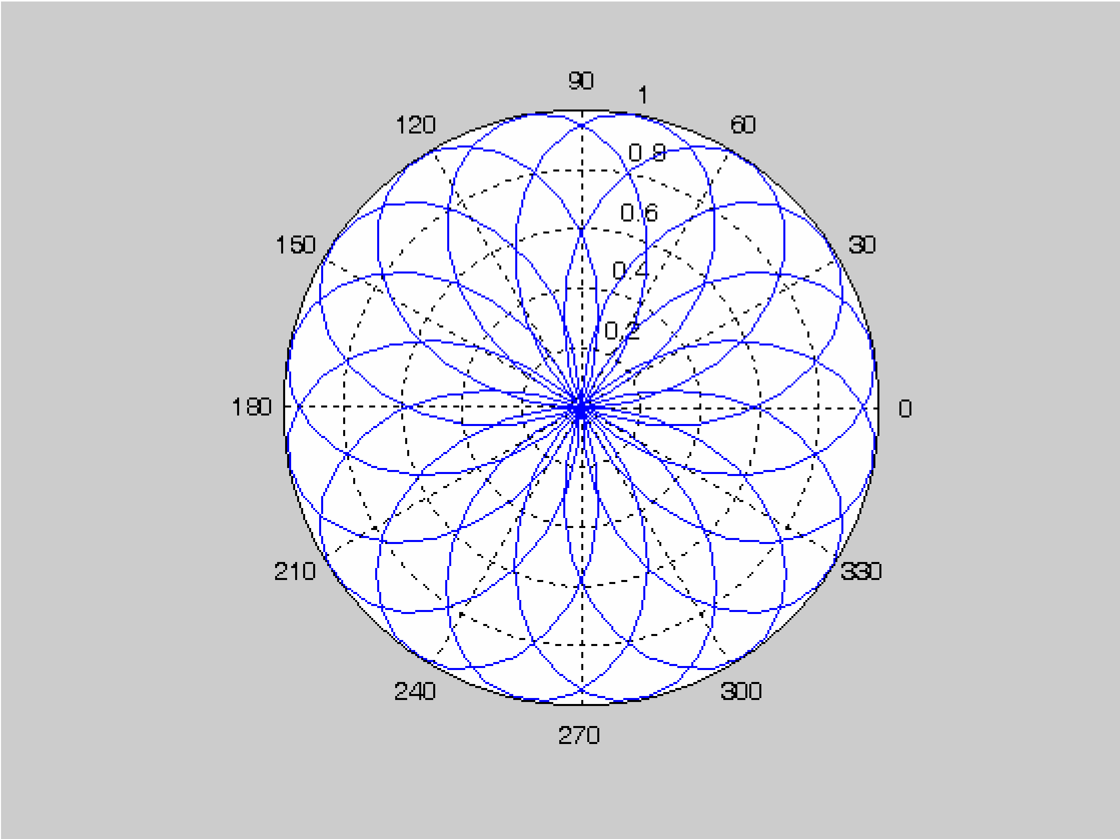
The smallest value of n for which $\frac{16n\pi}{5}$ is a multiple of 2π is $n = 5$.

so $0 \le \theta \le 10\pi$

Step 2

Use the following commands to get the graph

```
>>    theta=linspace(0,10*pi,300);
>>    r=sin(8*theta/5);
>>    polar(theta,r)
```



Example 11.1.18 Use Matlab to plot $r = 2 + 4\cos\theta$.

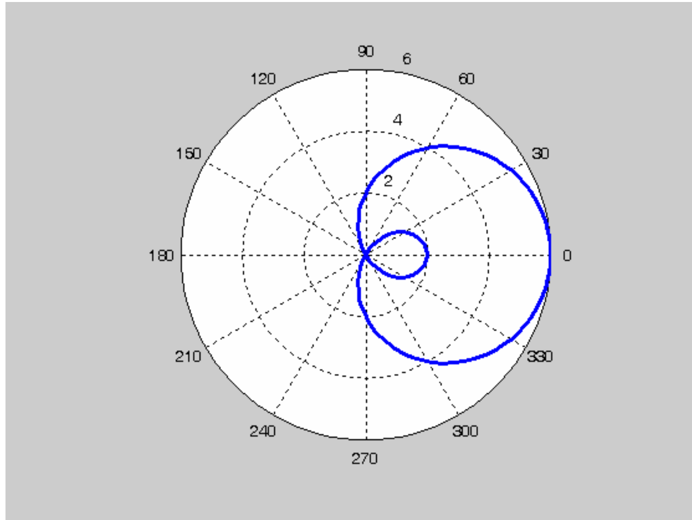
Solution

Step 1 Domain

so $0 \le \theta \le 2\pi$

Step 2

```
>>    theta=linspace(0,2*pi,100);
>>    r=2+4*cos(theta)
>>    polar(theta,r)
```



Example 11.1.19 Use Matlab to plot $r = 4 + 2\cos\theta$.

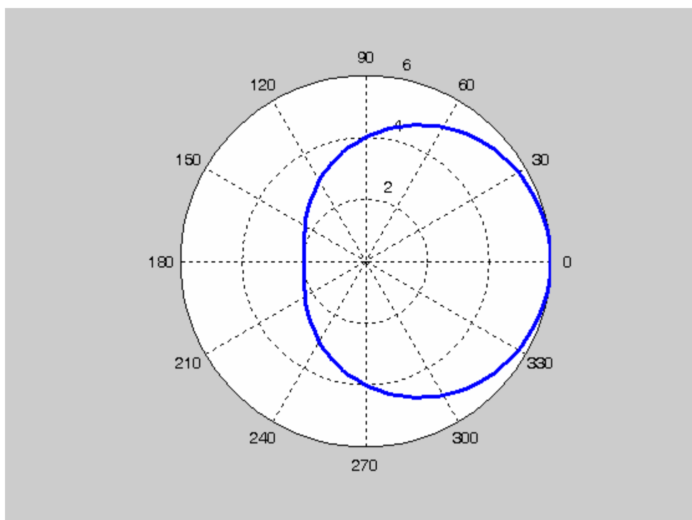
Solution

Step 1 Domain

so $0 \le \theta \le 2\pi$

Step 2

```
>>    theta=linspace(0,2*pi,100);
>>    r=4+2*cos(theta)
>>    polar(theta,r)
```



Exercise 11.1.20 Use Matlab to plot the following graphs

1. $r = \cos\left(\frac{3\theta}{2}\right)$
2. $r = 1 + \cos\theta$
3. $r = 0.5 + \cos\theta$
4. $r = 1.5 + \cos\theta$

Important polar graphs

Lines

1. $\theta = \alpha$

A line that goes through origin and makes an angle α with positive X-axis

2. $r\cos\theta = a$

In rectangular coordinates: $x = a$

Hence, a vertical line

3. $r\sin\theta = b$

In rectangular coordinates: $y = b$

Hence, a horizontal line

Circles

1. $r = a$ — circle of radius a

2. $r = 2a\cos\theta$
- circle of radius $|a|$ with centre at $(a, 0)$
- e.g. see graph of $r = 2\cos\theta$ in Example 11.1.8

3. $r = 2b\sin\theta$
- circle of radius $|b|$ with centre at $(0, b)$
- e.g. see graph of $r = 4\sin\theta$ in Exercise 11.1.9

Exercise 11.1.21 Plot

a. $r = 4$

b. $r = 4\cos\theta$

c. $r = -6\sin\theta$

Roses

$r = a\cos n\theta$ *OR* $r = a\sin n\theta$ (n: positive integer)

n-even
rose with $2n$ leaves

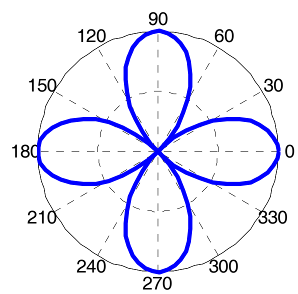
See graph of $r = \cos 2\theta$ in Example 11.1.11 which had 4 leaves

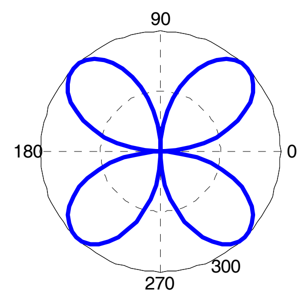
n-odd
rose with n leaves

See graph of $r = 2\sin 5\theta$ in Example 11.1.16 which had 5 leaves

Observe the following graphs

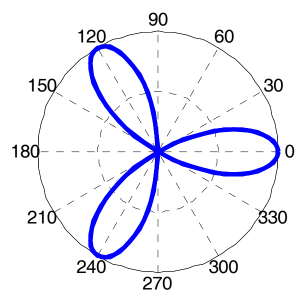
$r = \cos 2\theta$

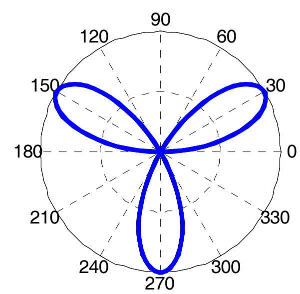




$r = \sin 2\theta$

$r = \cos 3\theta$





$r = \sin 3\theta$

- n-even
  Symmetric about X-axis, Y-axis and pole
- n-odd
  - $\cos n\theta$: Symmetric about X-axis
  - $\sin n\theta$: Symmetric about Y-axis

Cardioids

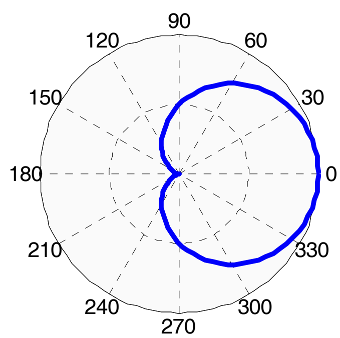
$$r = a \pm a\cos\theta = a(1 \pm \cos\theta)$$
$$r = a \pm a\sin\theta = a(1 \pm \sin\theta)$$

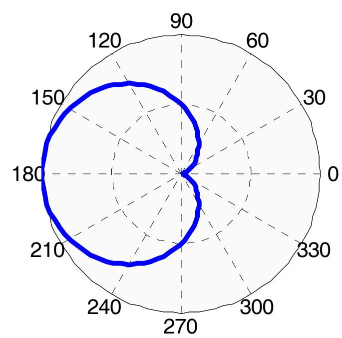
- heart shaped
- pass through pole

See graph of $r = 1 + \cos\theta$ in Example 11.1.10

Observe the following graphs of cardioids

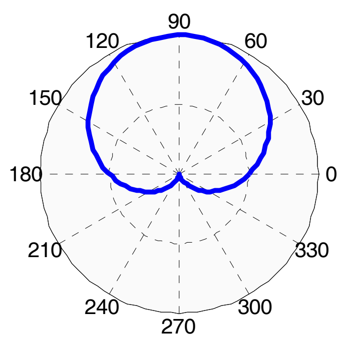
$r = a(1+\cos\theta)$

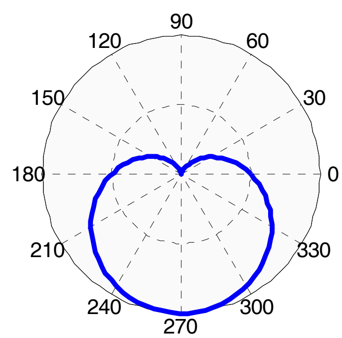




$r = a(1-\cos\theta)$

$r = a(1+\sin\theta)$





$r = a(1-\sin\theta)$

Limacons

$r = a \pm b\cos\theta$

$r = a \pm b\sin\theta$

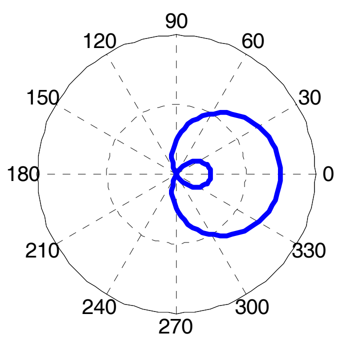
$(a < b)$

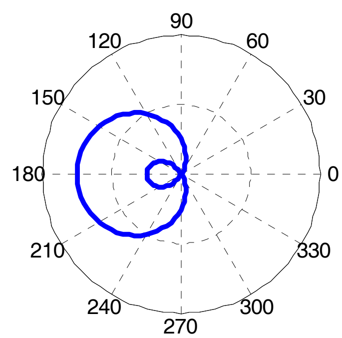
- have an inner loop
- pass through pole

See graph of $r = 2 + 4\cos\theta$ in Example 11.1.18

Observe the following graphs

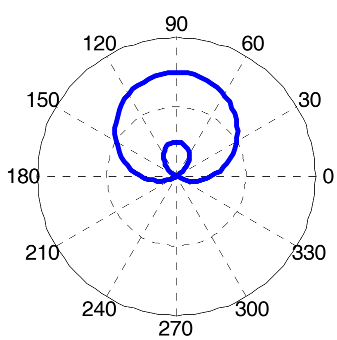
$r = 1 + 2\cos\theta$

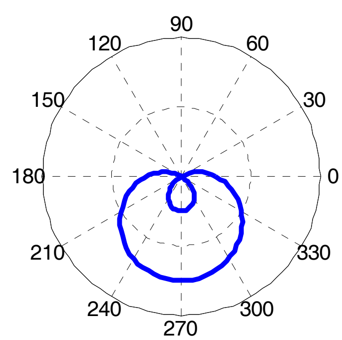




$r = 1 - 2\cos\theta$

$r = 1 + 2\sin\theta$





$r = 1 - 2\sin\theta$

Limacons

$r = a \pm b\cos\theta$

$r = a \pm b\sin\theta$

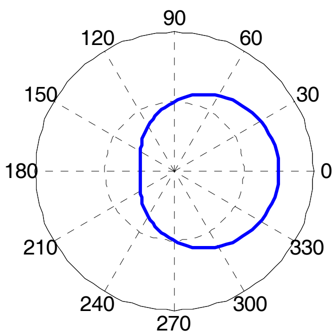
$(a > b)$

- do not have inner loop
- do not pass through pole

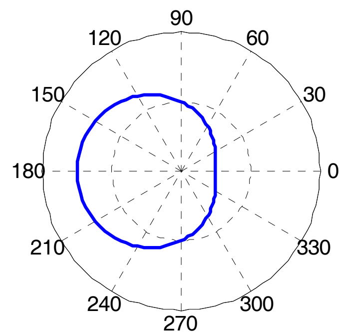
See graph of $r = 4 + 2\cos\theta$ in Example 11.1.19

Observe the following graphs

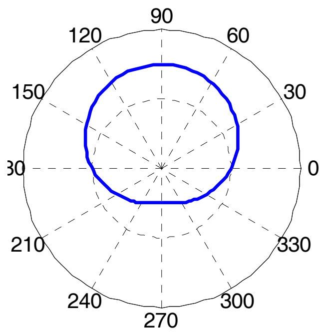
$r = 2 + \cos\theta$



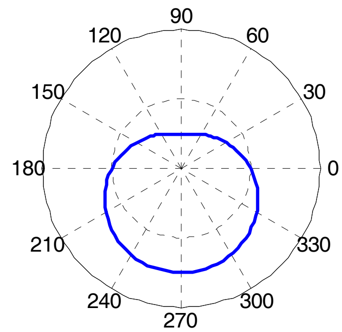
$r = 2 - \cos\theta$



$r = 2 + \sin\theta$



$r = 2 - \sin\theta$



Exercise 11.1.22 Plot $r = 1.25 + \cos\theta$, $r = 1.5 + \cos\theta$, $r = 2 + \cos\theta$ and compare. Next note the observations explained for Figure 11.1.20 in the book.

Lemniscates

$r^2 = a\cos 2\theta$

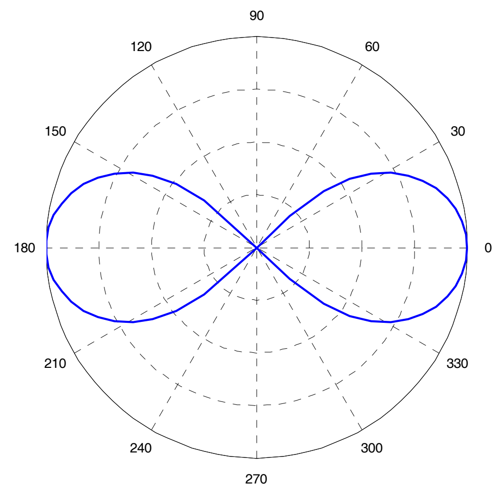
$r^2 = a\sin 2\theta$

- graph with two leaves
- pass through pole

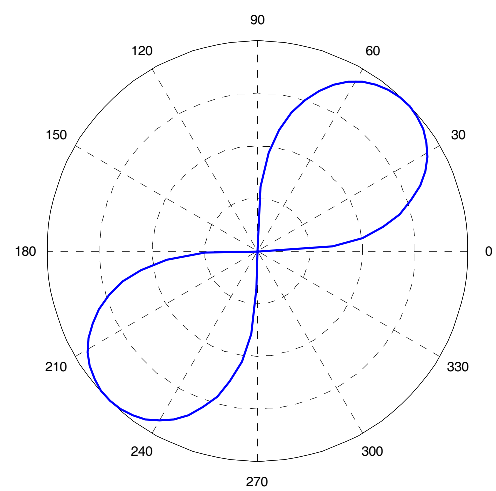
See Exercise 11.1.14

Observe the following graphs

$r^2 = 4\cos 2\theta$

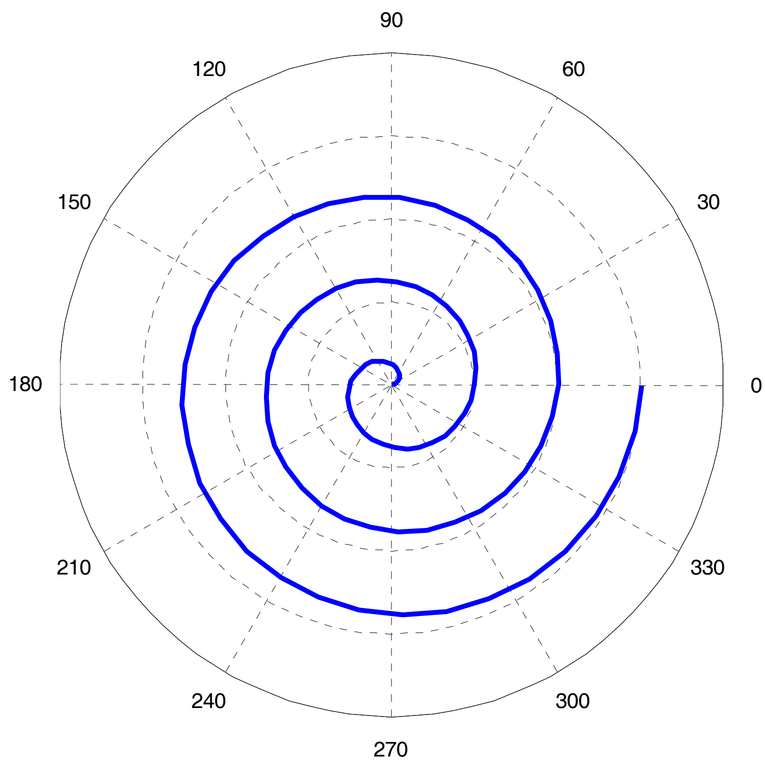


$r^2 = 4\sin 2\theta$



Spirals

$$r = a\theta$$



End of 11.1

Do Qs: 1-59