Section 2.4 *The precise definition of a limit*

(only problems like Examples 1, 2, 3 are in the syllabus)

Learning outcomes

After completing this section, you will inshaAllah be able to

1. understand the precise definition of limit
2. use the definition of limit to study limits of some functions

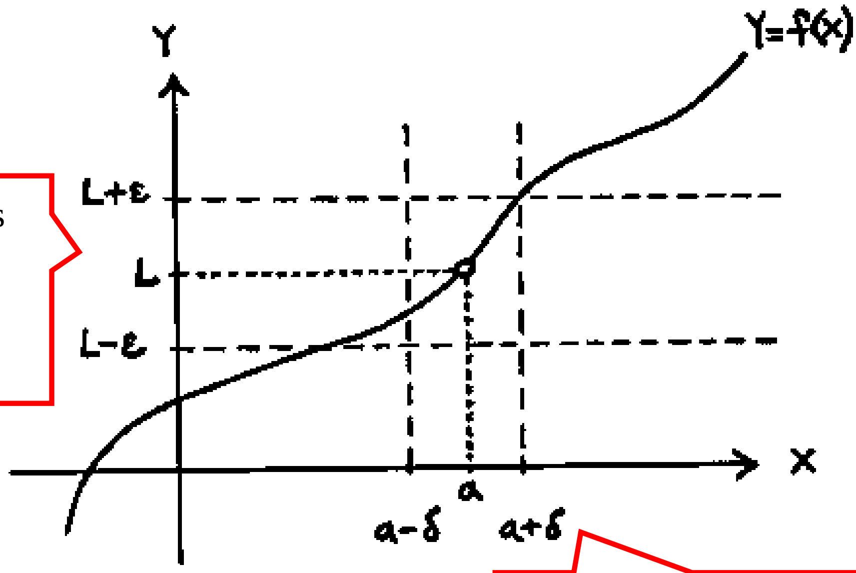
Formal definition of limit

- Look at example of Page 2.2_2 and recall from Page 2.3_2.

> Informally $\lim_{x \to a} f(x) = L$
>
> if we can make $f(x)$ as close to L as we like
>
> by taking x sufficiently close to a

- Formally we state this as

> Formally $\lim_{x \to a} f(x) = L$ if
>
> for every given number $\varepsilon > 0$, we can find a number $\delta > 0$ such that
>
> $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$



Recall $|f(x) - L| < \varepsilon$ means

$$-\varepsilon < f(x) - L < \varepsilon$$

or $$L - \varepsilon < f(x) < L + \varepsilon$$

Similarly $|x - a| < \delta$ means

$$-\delta < x - a < \delta$$

or $$a - \delta < x < a + \delta$$

See example 1 done in class

Strategy for proving $\lim_{x \to a} f(x) = L$ using definition

What to do?

- Consider any number $\varepsilon > 0$.
- We are required to show that we can find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \varepsilon$$

This is usually done through following two main steps

- Analyzing $|f(x) - L| < \varepsilon$ to make a choice for δ.

  This involves starting with the expression $|f(x) - L|$ and simplifying it reach the expression involving $|x - a|$.

- Using the chosen δ to formally prove the limit by showing that

$$0 < |x - a| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \varepsilon$$

See examples 2, 3 done in class

Do exercises given in class

Formal definition of right sided limit

- From the previous discussion it is easy to see the following definition

Formally $\lim_{x\to a^+} f(x) = L$ if

for every given number $\varepsilon > 0$, we can find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } a < x < a + \delta$$

i.e.

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Strategy for proving $\lim_{x\to a^+} f(x) = L$ using definition

What to do?

- Consider any number $\varepsilon > 0$.
- We are required to show that we can find a number $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

This is usually done through following two main steps

- Analyzing $|f(x) - L| < \varepsilon$ to make a choice for δ.

  This involves starting with the expression $|f(x) - L|$ and simplifying it reach the expression involving $x - a$.

- Using the chosen δ to formally prove the limit by showing that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

See example 4 done in class

Formal definition of left sided limit

- Similar to the definition of right sided limit we can write
- The formal definition of left sided limit is similar to the definition of right sided limit and can be understood exactly in similar manner.

Read page 118 of the book

End of 2.4