King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2008-2009 (082)

Monday, March 30, 2009 **Allowed Time: 2 hours**

Name: KEY

ID Number: ____________

Section Number:__________ **Serial Number:**__________

Instructions:
1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work. No points for answers without justification.**
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 9 different problems (5 pages + cover page)

Problem No	Grade	Maximum Points
1		14
2		24
3		8
4		8
5		6
6		8
7		16
8		8
9		8
Total		100

1. (a) [**3 points**] Write the following statement as a limit:

"$f(x)$ increases without bound as x approaches a from the left".

$$\lim_{x \to a^-} f(x) = +\infty$$

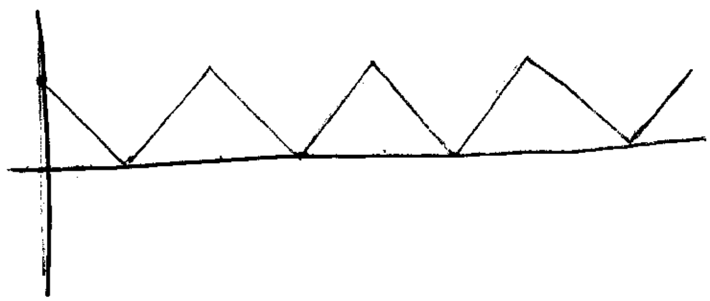
$\boxed{0 \text{ or } 3 \text{ pts}}$

(b) [**4 points**] TRUE or FALSE: "If f has a domain $[0, +\infty)$ and has no horizontal asymptote, then $\lim_{x \to +\infty} f(x) = +\infty$ or $\lim_{x \to +\infty} f(x) = -\infty$".

[If TRUE, state the reason. If FALSE, illustrate graphically].

1 False. Take a periodic function f

3



(c) [**7 points**] Sketch the graph of a function f that satisfies the following conditions:

1 point for each part

i. $f(-1) = 3$

ii. $\lim_{x \to -1^-} f(x) = 4$

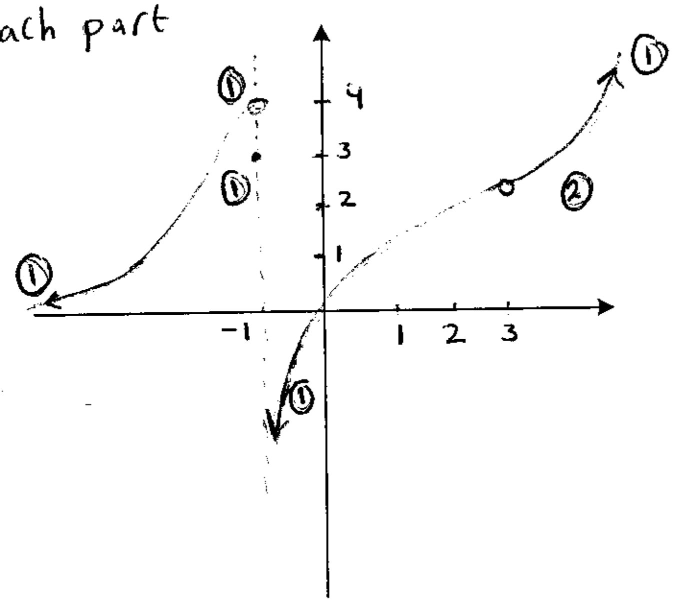
iii. $\lim_{x \to -1^+} f(x) = -\infty$

iv. $f(3)$ is undefined

v. $\lim_{x \to 3} f(x) = 2$

vi. $\lim_{x \to +\infty} f(x) = +\infty$

vii. $\lim_{x \to -\infty} f(x) = 0$



other graphs are possible

2. Find the limit if it exists.

(a) [**6 points**] $\lim_{x\to -4} \frac{x^3-16x}{x+4}$ $= \lim_{x\to -4} \frac{x(x-4)(x+4)}{x+4}$ 3

$= \lim_{x\to -4} x(x-4)$ 2

$= 32$ 1

(b) [**6 points**] $\lim_{x\to 12} \frac{|12-x|}{x-12}$

2 • $\lim_{x\to 12^-} \frac{|12-x|}{x-12} = \lim_{x\to 12^-} \frac{12-x}{x-12} = \lim_{x\to 12^-} -1 = -1$

2 • $\lim_{x\to 12^+} \frac{|12-x|}{x-12} = \lim_{x\to 12^+} \frac{-(12-x)}{x-12} = \lim_{x\to 12^+} 1 = 1$

2 Since $\lim_{x\to 12^-} \frac{|12-x|}{x-12} \neq \lim_{x\to 12^+} \frac{|12-x|}{x-12}$, then $\lim_{x\to 12} \frac{|12-x|}{x-12}$ does not exist

(c) [**6 points**] $\lim_{x\to 3} g(x)$, where $2x-1 \le g(x) \le x^2-5x+11$

Since $\lim_{x\to 3} 2x-1 = 5$ 2

& $\lim_{x\to 3} x^2-5x+11 = 5$, 2

then by the Squeeze Theorem, 1

$\lim_{x\to 3} g(x) = 5$ 1

(d) [**6 points**] $\lim_{x\to 6^+} \tan^{-1}(\ln(x-6))$

as $x \to 6^+$, $\ln(x-6) \longrightarrow -\infty$ 3

So $\tan^{-1}(\ln(x-6)) \longrightarrow -\frac{\pi}{2}$ 3

Thus $\lim_{x\to 6^+} \tan^{-1}(\ln(x-6)) = -\frac{\pi}{2}$

3. [**8 points**] Using the ϵ, δ definition of limit, prove that $\lim_{x\to 1}\left(-1+\frac{3}{2}x\right)=\frac{1}{2}$

3 Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$\left|\left(-1+\tfrac{3}{2}x\right)-\tfrac{1}{2}\right| < \varepsilon \quad \text{whenever} \quad 0 < |x-1| < \delta.$$

But $\left|\left(-1+\frac{3}{2}x\right)-\frac{1}{2}\right| = \left|\frac{3}{2}x-\frac{3}{2}\right| = \frac{3}{2}|x-1|$. Thus, we want

3 $$\tfrac{3}{2}|x-1| < \varepsilon \quad \text{whenever} \quad 0<|x-1|<\delta$$

that is, $$|x-1| < \tfrac{2\varepsilon}{3} \quad \text{whenever} \quad 0<|x-1|<\delta.$$

2 Thus, we may choose $\delta = \frac{2\varepsilon}{3}$.

Note: No need to check that $\delta = \frac{2\varepsilon}{3}$ works.

4. [**8 points**] Let $f(x) = \begin{cases} \sqrt{x+2} & \text{if } -2 \le x \le 2 \\ x^3-2x & \text{if } x > 2. \end{cases}$ Is f continuous at $x = 2$. If not, what kind of discontinuity does f have at $x = 2$. Justify your answers.

2 • $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3-2x = 8-4=4$

2 • $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{x+2} = \sqrt{4} = 2$

1 Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, then $\lim_{x\to 2} f(x)$ does not exis &

hence f is not continuuous at $x=2$.

2 Since $\lim_{x\to 2^+} f(x)$ exists, $\lim_{x\to 2^+} f(x)$ exists & $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$,

1 then f has a jump discontinuity at $x=2$

5. [**6 points**] Where is the function $f(x) = \dfrac{1}{3-\sqrt{x}}$ continuous? f is continuous in its domain

2 • Because of the term $\sqrt{x}$, we must have $x \geq 0$.

2 • We also must have $3-\sqrt{x} \neq 0$:

$$3-\sqrt{x} = 0 \implies \sqrt{x} = 3 \implies x = 9.$$

So we must have $x \neq 9$

2 f is continuous in $[0, 9) \cup (9, +\infty)$

6. [**8 points**] Show that the equation $e^{-x} = 2 - x$ has a root in the interval $(1, 2)$.

1 Apply the Intermediate Value Theorem by letting

1 $f(x) = e^{-x} + x - 2$, $[a,b] = [1,2]$, $N = 0$

2 . f is continuous on $[1,2]$

1 . $f(1) = e^{-1} + 1 - 2 = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$

1 . $f(2) = e^{-2} + 2 - 2 = e^{-2} > 0$

So $N = 0$ is between $f(1)$ & $f(2)$. Then by the IVT, there is a number c in $(1,2)$ such that

2 $f(c) = 0$,

that is $e^{-c} + c - 2 = 0$

or $e^{-c} = 2 - c$.

7. (a) [**8 points**] Find $\lim_{x\to+\infty}(\sqrt{x^2+1} - x)$. $\infty - \infty$, undefined

$$= \lim_{x\to+\infty} \sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} \quad 2$$

$$= \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1} + x} \quad 1$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1} + x}{x}} \quad 2$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}} + 1} \quad 1$$

$$= \frac{0}{1+1} = 0 \quad 2$$

(b) [**8 points**] Find the horizontal asymptotes of $f(x) = e^{x-x^2}$.

We find $\lim_{x\to+\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$.

Let $u = x - x^2$. Then

. $\lim_{x\to+\infty} u = \lim_{x\to+\infty} x - x^2 = \lim_{x\to+\infty} x^2(\frac{1}{x} - 1) = +\infty(0-1) = -\infty$ 2

. $\lim_{x\to-\infty} u = \lim_{x\to-\infty} x - x^2 = \lim_{x\to-\infty} x^2(\frac{1}{x} - 1) = +\infty(0-1) = -\infty$ 2

So $\lim_{x\to+\infty} f(x) = \lim_{u\to-\infty} e^u = 0$ 1

& $\lim_{x\to-\infty} f(x) = \lim_{u\to-\infty} e^u = 0$ 1

8. **[8 points]** Find an equation of the tangent line to the curve $y = \dfrac{1}{x^2 - x}$ at the point $\left(2, \dfrac{1}{2}\right)$. [You must use limits]

• Slope $= m = \lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ 2

$= \lim_{x \to 2} \dfrac{\frac{1}{x^2-x} - \frac{1}{2}}{x-2}$

$= \lim_{x \to 2} \dfrac{2 - x^2 + x}{2(x^2-x)(x-2)} = \lim_{x \to 2} \dfrac{-(x-2)(x+1)}{2x(x-1)(x-2)}$

$= \lim_{x \to 2} \dfrac{-(x+1)}{2x(x-1)}$ 2 $= -\dfrac{3}{4}$ 1

• An equation for the tangent line is

$y - \frac{1}{2} = -\frac{3}{4}(x-2)$ 3

$\Rightarrow \quad y = -\frac{3}{4}x + 2$

9. The displacement (in meters) of a particle moving in a straight line is given by the equation $s(t) = 3t^2 - 4t + 1$, where t is measured in seconds.

(a) **[2 points]** Find the average velocity over the time interval $[0, 3]$.

$v_{ave} = \dfrac{s(3) - s(0)}{3 - 0}$ 1

$= \dfrac{16 - 1}{3}$

$= \dfrac{15}{3}$ 1

(b) **[6 points]** Use limits to find the instantaneous velocity when $t = 2$.

$v(2) = \lim_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ 2

$$\left.\begin{aligned} &= \lim_{h \to 0} \frac{3(2+h)^2 - 4(2+h) + 1 - 5}{h} \\ &= \lim_{h \to 0} \frac{8h + 3h^2}{h} \\ &= \lim_{h \to 0} 8 + 3h \end{aligned}\right\} 2$$

$= 8 \quad m/s \quad \} \; 2$

$\boxed{7\,(a)}$ Another Solution

$$\lim_{x\to+\infty} \left(\sqrt{x^2+1} - x\right) \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \quad ②$$

$$= \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}+x} \quad ①$$

$$= 0 \quad ③ \qquad \text{Since } \lim_{x\to+\infty} \sqrt{x^2+1}+x = \infty \quad ②$$

$\boxed{8}$ Slope $= m = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$ ②

$$= \lim_{h\to 0} \frac{\frac{1}{(2+h)^2-(2+h)} - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{-(h+3)}{2(h^2+3h+2)} \quad ②$$

$$= \frac{-3}{4} \quad ①$$

an equation of the tangent line is

$$y - \frac{1}{2} = -\frac{3}{4}(x-2) \quad ③$$

$$\Rightarrow \quad y = -\frac{3}{4}x + 2$$