**King Fahd University of Petroleum and Minerals**
**Department of Mathematics and Statistics**
**Math 101- Calculus I**
**Exam I**
**2008-2009 (081)**

**Monday, November 10, 2008** **Allowed Time: 2 hours**

**Name:** ______________________

KEY

**ID Number:** ______________

**Section Number:** ______________ **Serial Number:** ______________

**Instructions:**

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 13 different problems (7 pages + cover page)

| Page # | Points | Maximum Points |
|---|---|---|
| 1 | | 17 |
| 2 | | 15 |
| 3 | | 14 |
| 4 | | 14 |
| 5 | | 17 |
| 6 | | 11 |
| 7 | | 12 |
| Total | | 100 |

1. (10 points) Find each of the following limits of the function $f$ whose graph is given in the adjacent figure

(a) $\lim_{x\to -2^-} f(x) = -\infty$

(b) $\lim_{x\to -1} f(x) = 0$

(c) $\lim_{x\to 0^-} f(x) = \infty$

(d) $\lim_{x\to 0^+} f(x) = -\infty$

(e) $\lim_{x\to 1} f(x) = 2$

(f) $\lim_{x\to 2^-} f(x) = 3$

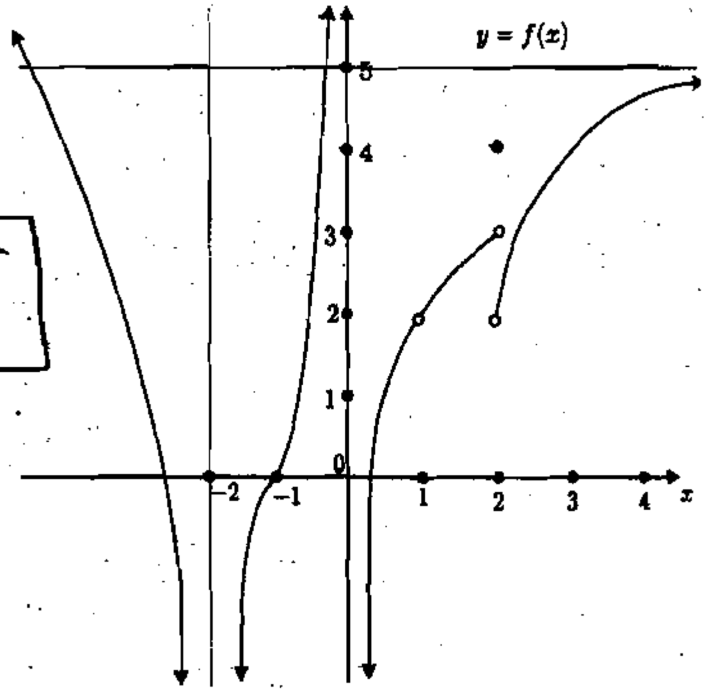
(g) $\lim_{x\to 2^+} f(x) = 2$

(h) $\lim_{x\to 2} f(x) =$ DNE

(i) $\lim_{x\to -\infty} f(x) = \infty$

(j) $\lim_{x\to +\infty} f(x) = 5$

1 point each




2. (7 points) Sketch the graph of an example of a function $f$ that satisfies the following conditions:

(a) $f'(-3) = f'(3) = 0$,

(b) $\lim_{x\to 0^-} f(x) = -1$,

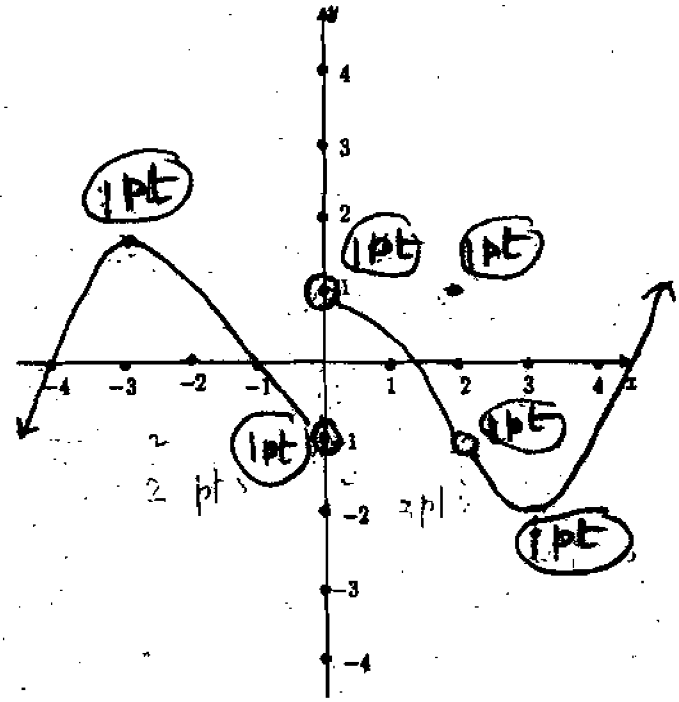
(c) $\lim_{x\to 0^+} f(x) = 1$,

(d) $f(0)$ is undefined, → 1pt

(e) $\lim_{x\to 2} f(x) = -1$,

(f) $f(2) = 1$.

Other graphs are possible



3. Evaluate each of the following limits (show your steps).

(a) (3 points) $\lim_{x\to 2} \frac{x^2-3x+2}{2-x} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(2-x)}$ 1 pt

$= \lim_{x\to 2} -(x-1)$ 1 pt

$= -1$ 1 pt

(b) (4 points) $\lim_{x\to+\infty} \frac{1-x-2x^3}{x^3+2x^2+1} = \lim_{x\to\infty} \frac{\frac{1-x-2x^3}{x^3}}{\frac{x^3+2x^2+1}{x^3}}$ 1 pt

$= \lim_{x\to\infty} \frac{\frac{1}{x^3}-\frac{1}{x^2}-2}{1+\frac{2}{x}+\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\left(\frac{1}{x^3}-\frac{1}{x^2}-2\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x}+\frac{1}{x^3}\right)}$ 2 pts

$= \frac{0-0-2}{1+0+0} = -2$ 1 pt

(c) (4 points) $\lim_{x\to-\infty} \frac{\sqrt{3x^2+7}}{4x-11} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{3x^2+7}}{\frac{1}{x}(4x-11)}$ 1 pt

$= \lim_{x\to-\infty} \frac{-\frac{1}{\sqrt{x^2}}\sqrt{3x^2+7}}{4-\frac{11}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{3+\frac{7}{x^2}}}{4-\frac{11}{x}}$ 2 pts

$= -\frac{\sqrt{3}}{4}$. 1 pt

(d) (4 points) $\lim_{x\to\frac{1}{2}^-} \frac{12x^2-6x}{|2x-1|} = \lim_{x\to\frac{1}{2}^-} \frac{6x(2x-1)}{-(2x-1)}$ 2 pts

$= \lim_{x\to\frac{1}{2}^-} -6x = -3$. 2 pts

4. (4 points) If $\lim_{x\to 2} f(x) = 7$ and $\lim_{x\to 2} g(x) = 3$, find $\lim_{x\to 2} \frac{\sqrt{x+f(x)}}{|x-2|-(g(x))^2}$. Justify each step.

$$\lim_{x\to 2} \sqrt{x+f(x)} = \sqrt{\lim_{x\to 2}(x+f(x))} = \sqrt{2+7} = 3 \quad 1pt$$

$$\lim_{x\to 2} \left[|x-2|-(g(x))^2\right] = 0-9=-9 \quad 1pt$$

$$\Rightarrow \lim_{x\to 2} \frac{\sqrt{x+f(x)}}{|x-2|-(g(x))^2} = \frac{\lim_{x\to 2}\sqrt{x+f(x)}}{\lim(|x-2|-(g(x))^2)} \quad 1pt$$

$$= -\frac{3}{9} = -\frac{1}{3} \quad 1pt$$

5. (10 points) Use the Squeeze Theorem to show that $\lim_{x\to 0} \sin x \cdot \cos \frac{1}{x} = 0$.

We know that $-1 \leq \cos\frac{1}{x} \leq 1$ (A) 1pt

We discuss two cases:

Case 1: If $x \to 0^+$, then $\sin x > 0$ 1pt

(A) $\Rightarrow -\sin x \leq \sin x \cdot \cos\frac{1}{x} \leq \sin x$ 1pt

We know that $\lim_{x\to 0^+}(-\sin x) = \lim_{x\to 0^+} \sin x = 0$ 1pt

$\Rightarrow \lim_{x\to 0^+} \sin x \cdot \cos\frac{1}{x} = 0$ by the squeeze Th. 1pt

Case 2: If $x \to 0^-$, then $\sin x < 0$ 1pt

(A) $\Rightarrow -\sin x \geq \sin x \cdot \cos\frac{1}{x} \geq \sin x$ 1pt

But $\lim_{x\to 0^-}(-\sin x) = \lim_{x\to 0^+}(\sin x) = 0$ 1pt

$\Rightarrow \lim_{x\to 0^-} \sin x \cdot \cos\frac{1}{x} = 0$ by the squeeze Th. 1pt

Case 1 & Case 2 $\Rightarrow \lim_{x\to 0} \sin x \cdot \cos\frac{1}{x} = 0$. 1pt

6. The displacement (in meters) of a particle moving in a straight line is given by the equation $S = 40 + 16t^2$, where $t$ is measured in seconds.

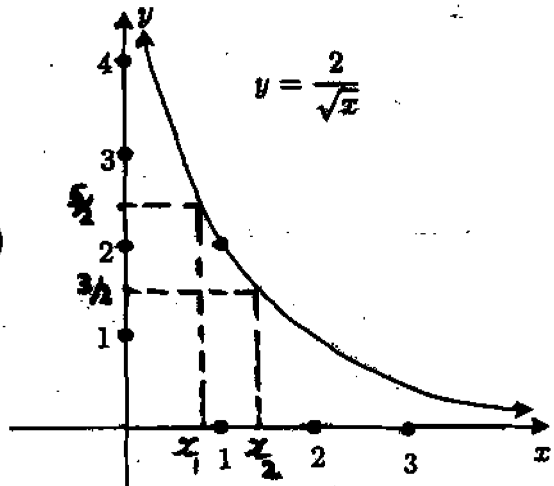
(a) (3 points) Find the average velocity of the particle over the time interval with endpoints between 1 and $1 + h$.

$$v_{ave} = \frac{1}{h}\left[S(1+h) - S(1)\right] \quad 1\text{ pt}$$
$$= \frac{1}{h}\left[40 + 16(1+h)^2 - (56)\right]$$
$$= \frac{1}{h}\left[40 + 16 + 32h + 16h^2 - 56\right]$$
$$= \frac{1}{h}\left[32h + 16h^2\right] \quad 1\text{ pt}$$
$$= (32 + 16h) \text{ m/sec} \quad 1\text{ pt}$$

(b) (2 points) Use part (a) to find the instantaneous velocity of the particle when $t = 1$.

$$v = \lim_{h\to 0} v_{ave}$$
$$= \lim_{h \to 0} (32 + 16h) = 32 \text{ m/sec} \quad 2\text{ pts}$$

7. (9 points) Use the graph of $f(x) = \frac{2}{\sqrt{x}}$ to find the largest a number $\delta$ such that $|f(x) - 2| < \frac{1}{2}$ whenever $0 < |x - 1| < \delta$. (Show your steps and write your answer in a rational form $\frac{p}{q}$).




2 pts

Let $x_1$ and $x_2$ as shown in the figure $\Longrightarrow$

$$\frac{5}{2} = \frac{2}{\sqrt{x_1}} \Rightarrow x_1 = \frac{16}{25} \quad 2\text{ pts}$$

and $$\frac{3}{2} = \frac{2}{\sqrt{x_2}} \Rightarrow x_2 = \frac{16}{9} \quad 2\text{ pts}$$

The largest value of $\delta$ =

The smallest of $(1 - x_1, x_2 - 1)$ 1 pt

= The smallest of $\left(\frac{9}{25}, \frac{7}{9}\right)$ 1 pt

$$= \frac{9}{25} \quad 1\text{ pt}$$

8. (8 points) Find an equation of the tangent line to the curve $f(x) = \frac{2}{x+3}$ at the point where $x = -1$. [You must use limits].

The slope of the required tangent line $= f'(-1)$ (1 pt)

We have $f'(-1) = \lim_{h \to 0} \frac{1}{h}[f(-1+h) - f(-1)]$ (1 pt)

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2}{2+h} - 1\right]$ (1 pt)

$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{2+h}\right)$ (1 pt)

$= \lim_{h \to 0} \frac{-1}{2+h} = -\frac{1}{2}$ (2 pts)

$\Rightarrow$ An equation of the tangent line at $(-1, 1)$. (1 pt)

is $y - 1 = -\frac{1}{2}(x+1)$ (1 pt)

9. (9 points) If $[x]$ denotes the greatest integer less than or equal to $x$, find all values of $x$ for which the following function is continuous:

$$f(x) = \begin{cases} [x], & \text{if } -2 \le x < 0 \\ x, & \text{if } 0 \le x < 1 \\ 3x - 2, & \text{if } 1 \le x \le 2 \end{cases} = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 3x-2, & 1 \le x \le 2 \end{cases}$$ (1 pt)

(Use limits to justify your answers).

$\Rightarrow \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -2 = -2$, $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} -1 = -1$, (2 pts)

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1$, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$, (2 pts)

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x-2) = 1 = f(1)$ (2 pts)

$\Rightarrow f$ is continuous on $[-2,-1) \cup (-1,0) \cup (0,2]$ (2 pts)

10. (6 points) Determine whether the function

$$f(x) = \frac{\sqrt{2x+9} - \sqrt{x+9}}{2x}$$

has a removable discontinuity, a jump discontinuity, or an infinite discontinuity at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(\sqrt{2x+9} - \sqrt{x+9})(\sqrt{2x+9} + \sqrt{x+9})}{2x(\sqrt{2x+9} + \sqrt{x+9})} \quad \text{1 pt}$$

$$= \lim_{x \to 0} \frac{(2x+9) - (x+9)}{2x(\sqrt{2x+9} + \sqrt{x+9})} \quad \text{1 pt}$$

$$= \lim_{x \to 0} \frac{x}{2x(\sqrt{2x+9} + \sqrt{x+9})} \quad \text{1 pt}$$

$$= \lim_{x \to 0} \frac{1}{2(\sqrt{2x+9} + \sqrt{x+9})} \quad \text{1 pt}$$

$$= \frac{1}{(2)(6)} = \frac{1}{12}. \quad \text{1 pt}$$

$\Rightarrow$ f has a removable discontinuity at 0 since f is discontinuous at 0 and $\lim_{x \to 0} f(x) = \frac{1}{12}$ } 1 pt

11. (5 points) Use the Intermediate Value Theorm to show that there is a root of the equation $x^6 + x^4 - 1 = 0$ in the interval $[-1, 1]$.

Let $f(x) = x^6 + x^4 - 1 \Rightarrow f$ is continuous on the interval $[-1,1]$. } 1 pt

We have $f(0) = -1$, and $f(1) = 1$. 2 pts

Since $f(0) < 0 < f(1) \Rightarrow$ there is a number $c$ in $(0,1)$ such that $f(c) = 0$ by the Intermediate Value Theorem $\Rightarrow$ There is at least one root of the given equation in the interval $[-1,1]$. } 2 pts

12. (4 points) The limit $\lim_{x\to\frac{\pi}{2}} \frac{6(\sin x - 1)}{2x-\pi}$ represents the derivative of some function $f$ at some number $a$. State such an $f$ and $a$. (give a reason to your answer)

$$\lim_{x\to\pi/2} \frac{6(\sin x - 1)}{2x-\pi} = \lim_{x\to\frac{\pi}{2}} \frac{6(\sin x - \sin\frac{\pi}{2})}{2(x-\pi/2)} \quad \text{1 pt}$$

$$= \lim_{x\to\pi/2} \frac{3\sin x - 3\sin\frac{\pi}{2}}{x - \pi/2} \quad \text{1 pt}$$

$$= f'(\pi/2) \quad \text{1 pt}$$

where $f(x) = 3\sin x$ and $a = \pi/2$. 1 pt

13. (8 points) Find the equations of all horizontal asymptotes to the graph of $f(x) = \tan^{-1}(e^{-2x} - 1)$. (Show your work)

Let $u = e^{-2x} - 1 \Rightarrow$

$\lim_{x\to\infty} u = -1$ and $\lim_{x\to-\infty} u = \infty$ 2 pts

$$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{u\to-1} \tan^{-1} u = -\frac{\pi}{4} \quad \text{2 pts}$$

$$\text{and} \quad \lim_{x\to-\infty} f(x) = \lim_{u\to\infty} \tan^{-1} u = \frac{\pi}{2} \quad \text{2 pts}$$

$\Rightarrow$ $y = -\frac{\pi}{4}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes of the graph of $f$. 2 pts