Extended reciprocal zeta function and an alternate formulation of the Riemann hypothesis

By
M. Aslam Chaudhry

Department of Mathematical Sciences,
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia.
E-mail: maslam@kfupm.edu.sa

Abstract

We define an extension of the reciprocal of the zeta function. Some properties of the function are discussed. We prove an alternate criteria for the proof of the Riemann hypothesis and the simplicity of the zeros of the zeta function.

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1. Introduction

Riemann proved that the zeta function (see [1, 2, 3, 6, 7]),

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s = \sigma + i\tau, \sigma > 1), \tag{1.1}$$

has a meromorphic continuation to the complex plane. This satisfies the functional equation (see [6], p.13 (2.1.1))

$$\zeta(s) = 2(2\pi)^{-(1-s)} \cos(\frac{\pi}{2}(1-s))\Gamma(1-s)\zeta(1-s), \tag{1.2}$$

and has simple zeros at $s = -2, -4, -6, \ldots$ called the *trivial zeros*. All the other zeros, called the *non-trivial zeros*, of the function are symmetric about the *critical line* $\sigma = \frac{1}{2}$ in the *critical strip* $0 \le \sigma \le 1$. The multiplicity of these non-trivial zeros (in general) is not known. Riemann conjectured that the non-trivial zeros of the function lie on the critical line $\sigma = \frac{1}{2}$. This conjecture is called the Riemann hypothesis. The zeta function has the integral representation ([6], p.18 (2.4.1))

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} dt}{e^t - 1} \qquad (\sigma > 1). \tag{1.3}$$

Writing the Möbius function as $\mu(n) = (-1)^k$ if the $n = p_1 p_2 \ldots p_k$ is a product of k distinct primes and zero otherwise. It is known that (see [3], p. 260)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (\sigma \geq 1). \tag{1.4}$$

The extension of the identity (1.4) to the region $1/2 < \sigma < 1$ would prove the Riemann hypothesis. There are several extensions of the zeta function. The zeta function belongs to a wider class of L-functions (see [2, 3, 6]). One of the well known extension of the zeta function is the Hurwitz zeta function ([6], p. 36(2.17))

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \qquad (s = \sigma + i\tau, \sigma > 1), \tag{1.5}$$

which has the integral representation([6], p.37(2.17.1))

$$\zeta(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-xt} dt}{1-e^{-t}} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(x-1)t} dt}{e^t - 1} \qquad (\sigma > 1, x > 0). \tag{1.6}$$

It follows from (1.1) and (1.5) that

$$\zeta(s,1) := \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \qquad (\sigma > 1), \tag{1.7}$$

leading to the fact that the Hurwitz zeta function generalizes the Riemann zeta function in the most natural way. It seems ironical that the function $1/\zeta(s,x)$ does not seem to be a useful extension of the reciprocal $1/\zeta(s)$ of the zeta function. We introduced a function that seems to be a natural generalization of the reciprocal $1/\zeta(s)$. Some properties of the function are discussed. We exploit the asymptotic representation of our function to give an alternate formulation of the Riemann hypothesis and simplicity of the zeros of the zeta function. For the other necessary and sufficient conditions of the Riemann hypothesis we refer to (see [6], section 14.32).

2. The Möbius inversion formula and applications to the Hurwitz zeta function

The Möbius inversion formula can be written in the form ([3], p.217)

$$g(x) = \sum_{n=1}^{\infty} f(nx) \Leftrightarrow f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx) \qquad (x > 0), \tag{2.1}$$

provided $\sum_{n=1}^{\infty} f(nx)$ and $\sum_{n=1}^{\infty} g(nx)$ both converge absolutely. The above formula can also be written in the form

$$g(x) = \sum_{n=1}^{\infty} f(\frac{x}{n}) \Leftrightarrow f(x) = \sum_{n=1}^{\infty} \mu(n) g(\frac{x}{n}) \qquad (x > 0). \tag{2.2}$$

Rewriting (1.5) we find that

$$x^s \zeta(s,x+1) := \sum_{n=1}^{\infty} \frac{1}{(1+\frac{n}{x})^s} \qquad (s = \sigma + i\tau, \sigma > 1), \tag{2.3}$$

which leads to a useful analytic representation ($x \geq 0$)

$$(1+x)^{-s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \zeta(s, 1+\frac{x}{n}) \qquad (s = \sigma + i\tau, \sigma > 1). \qquad (2.4)$$

The LHS in (2.4) is an entire function of s for all $x \geq 0$. In particular for $x = 0$ in (2.4) the classical identity (1.4) is recovered.

Since we have $(x/x+1)^s \to 1$ as $x \to \infty$, it follows from (2.4) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \zeta(s, 1+\frac{x}{n}) \sim x^{-s} \qquad (x \to \infty). \qquad (2.5)$$

Similarly we have $(1+x)^{-s} \to 1$ as $x \to 0^+$, it follows from (2.4) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \zeta(s, 1+\frac{x}{n}) \to 1 \qquad (x \to 0^+). \qquad (2.6)$$

3. The Extended reciprocal zeta function

A close look to the Hurwitz zeta functions shows that it is basically the extension of the series representation (1.1) of the zeta function obtained when n is replaced by $n+x+1$. We follow the same procedure in (1.4) and define the our extended reciprocal of the zeta function by

$$R(s,x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{(n+x)^s} \qquad (\sigma \geq 1,\ x \geq 0). \qquad (3.1)$$

The function $R(s,x)$ extends the reciprocal function $1/\zeta(s)$ in the most natural way as we have

$$R(s,0) = 1/\zeta(s) \qquad (\sigma \geq 1). \qquad (3.2)$$

The extension of the identity (3.2) to the region $1/2 < \sigma < 1$ should prove the Riemann hypothesis (see [3], p.261). An application of the Möbius inversion formula (2.2) in (3.1) leads to the relation

$$(\frac{x}{x+1})^s = \sum_{n=1}^{\infty} \frac{1}{n^s} R(s, \frac{1}{nx}) \qquad (\sigma \geq 1,\ x \geq 0). \qquad (3.3)$$

The relation (3.3) is important in the sense that it shows that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} R(s, \frac{1}{nx}) \sim 1 \qquad (x \to \infty). \qquad (3.4)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^s} R(s, \frac{1}{nx}) \sim x^s \qquad (x \to 0^+). \qquad (3.5)$$

Theorem The function $R(s,x)$ has the integral representation

$$R(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-xt} \Theta(t) dt \qquad (\sigma \geq 1, x \geq 0), \qquad (3.6)$$

where the function $\Theta(t)$ is defined by

$$\Theta(t) := \sum_{n=1}^{\infty} \mu(n) e^{-nt} \qquad (t > 0). \tag{3.7}$$

Proof Multiplying both sides in (3.7) by e^{-xt} and then taking the Mellin transform we find that

$$M[e^{-xt}\Theta(t); s] = \Gamma(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{(n+x)^s} = \Gamma(s) R(s,x) \qquad (\sigma > 1). \tag{3.8}$$

Dividing both sides in (3.8) by $\Gamma(s)$, as the gamma function does not vanish in the complex plane, leads to (3.6).

Corollary

$$R(s,0) = \frac{1}{\zeta(s)} \qquad (\sigma > 1) \tag{3.9}$$

Theorem The function $R(s,x)$ has the Taylor series representation at $x = 0$ given by

$$R(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)(-x)^n}{\zeta(s+n) n!} = \sum_{n=0}^{\infty} \frac{(s)_n (-x)^n}{\zeta(s+n) n!} \qquad (\sigma \geq 1,\ 0 \leq x < 1) \tag{3.10}$$

Proof Replacing the exponential function in (3.5) by its series expansion leads to the representation

$$R(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \int_0^{\infty} t^{s+n-1} \Theta(t) dt = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)(-x)^n}{\zeta(s+n) n!} \tag{3.11}$$

which is exactly (3.10).

Remark Since we have $\zeta(\infty) = 1$, the representation (3.10) may be viewed as the perturbation of the geometric series

$$(1+x)^{-s} = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} (-x)^n . \tag{3.12}$$

4. Alternate formulation of the Riemann hypothesis

Taking the Mellin transform in x of both sides of (3.7) and using *Ramanujan's formula* (see [3], p.218) we find that

$$R(s,x) = \sum_{k=0}^{\infty} \frac{(s)_k (-x)^k}{\zeta(s+k) k!} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)\zeta(s-z)} x^{-z} dz \tag{4.1}$$

where the line $z = c$ passes through the region of analyticity of the integrand. The poles $z = -n$ ($n = 0,1,2,...$) of the above integrand are to the LHS of the line of integration leading to the representation (3.11). The integrand in (4.1) has poles as well to the RHS of the line of integration at $z = s + n$ ($n = 0,1,2,...$) and that at $z = s - \rho$ where ρ's are the non-trivial zeros of the zeta function. Taking the sum over the residues leads to the asymptotic representation for large values of x to give

$$R(s,x) \sim \sum_{n=0}^{\infty} R_n x^{-s-n} + \sum_{\rho} \Re_{\rho} x^{\rho - s} \qquad (x \to \infty), \tag{4.2}$$

where, we define

$$R_n := x^{s+n}\,\mathrm{Res}[\frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)\zeta(s-z)}x^{-z};s+n], \tag{4.3}$$

$$\Re_\rho(x,s) := x^{s-\rho}\,\mathrm{Res}[\frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)\zeta(s-z)}x^{-z};s-\rho]. \tag{4.4}$$

It is to be noted that if the zero ρ of the zeta function is simple, $\Re_\rho$ does not depend on x and is just a function of s. Moreover, the contribution to the asymptotic due to the presence of several zeros of the zeta function on the line $\sigma=\sigma_M$ will not exceed $Cx^{\sigma_M+\varepsilon}$ ($\varepsilon>0$). However, if the zero ρ is of multiplicity N_ρ, then for each s, $\Re_\rho$ is a polynomial of degree $\le N_\rho-1$ in $\log x$. The asymptotic representation (4.2) shows, under the assumption of the simplicity of the zeros of the zeta function, that

$$R(s,x)\sim\sum_\rho \Re_\rho x^{\rho-s} = \mathrm{O}(x^{\sigma_M-\sigma+\varepsilon}) \qquad (x\to\infty,\ \sigma>1,\ \forall\varepsilon>0), \tag{4.5}$$

where $\sigma_M := \sup\{\mathrm{Re}(\rho):\zeta(\rho)=0\}$.The Riemann hypothesis is true and its zeros are simple if $\sigma_M = 1/2$. Hence it follows that the Riemann hypothesis is true and its zeros are simple if we have in particular for $s=3/2$

$$R(3/2,x)=O(x^{-1+\varepsilon}) \qquad (\forall\varepsilon>0,\ x\to\infty), \tag{4.6}$$

which provides an alternate formulation of the Riemann hypothesis and the simplicity of the zeros of the zeta function. We use Mathematica to plot the function in Figure I by taking the sum of the first ten thousand terms of the series (3.1) and find consistency with (4.6). However an analytic proof of (4.6) is needed to prove the Riemann hypothesis.

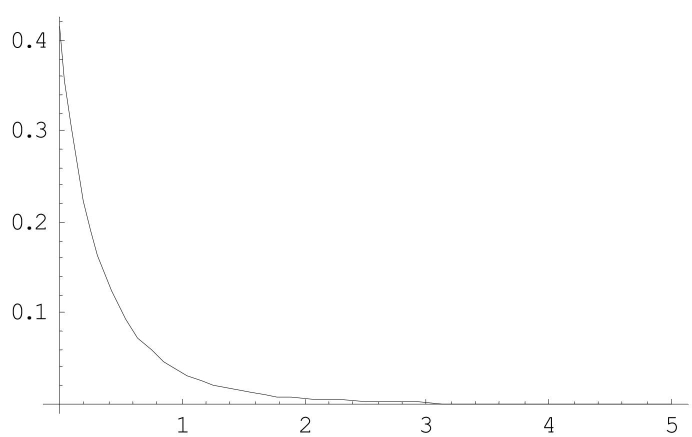

Figure I The graph of $R(s,x)$ for $s=1.5$ using (3.1) for large x.

5. Concluding Remarks

We have the series representation (see [4], p.357 (54.6.1))

$$\zeta(s,x+1)=\sum_{n=0}^{\infty}\frac{(s)_n\zeta(s+n)(-x)^n}{n!} \qquad (0<x<1), \tag{5.1}$$

for the Hurwitz zeta function. A comparison of the representations (3.10) and (5.1) is not without interest. The extended reciprocal zeta function extends the reciprocal zeta function in the most natural way and provides an alternate criteria for the proof of the Riemann hypothesis and simplicity of the zeros of the zeta function. An analytic proof of (4.6) will resolve the classical problem of the Riemann hypothesis and the simplicity of the zeros of the zeta function. Moreover, we have not been able to prove the functional equation for the extended reciprocal zeta function though it is expected that there should be a functional equation for the function similar to the one known for Hurwitz's zeta function.

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