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The asymptotic representation of some series and the Riemann hypothesis

By

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Abstract

We present a conjecture about the asymptotic representation of certain series. The conjecture implies the Riemann hypothesis and it would also indicate the simplicity of the non-trivial zeros of the zeta-function.

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1. Introduction

We present a conjecture about the asymptotic behavior of a class of functions represented by a series. A special case of the conjecture has applications in the theory of zeta functions ([1, 2, 3, 4, 5]).

Riemann [6] proved that the zeta-function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s = \sigma + i\tau, \sigma > 1), \tag{1.1}$$

has a meromorphic continuation to the complex plane which satisfies the functional equation ([5], p.13, eq. (2.1.1))

$$\zeta(s) = 2(2\pi)^{-(1-s)} \cos(\frac{\pi}{2}(1-s))\Gamma(1-s)\zeta(1-s) := \chi(s)\zeta(1-s). \tag{1.2}$$

The Riemann zeta function has simple zeros at $s = -2, -4, -6, \ldots$ called *trivial zeros*. All other zeros, called *non-trivial zeros*, of the zeta function are symmetric about the *critical line*

$\sigma = 1/2$ and are in the *critical strip* $0 \le \sigma \le 1$. The multiplicity of these non-trivial zeros (in general) is not known. Riemann conjectured that all non-trivial zeros of the zeta function lie on the critical line $\sigma = 1/2$. This conjecture is called the *Riemann Hypothesis* (RH, for short). We present a conjecture on the asymptotic behavior of a class of functions. It will be shown that the proof of a special case of the conjecture would imply the truth of RH and demonstrate the simplicity of the zeros of the zeta function. For definitions and terminology, we refer to [7, 8, 9].

2. A restricted result about the asymptotic representation of series

We start with a trivial observation. Let $u(x)$ ($x \ge x_0 > 0$) be a decreasing, continuous and positive function such that $\sum_{n=1}^{\infty}(-1)^{n-1}u(x+n)$ ($x \ge x_0 > 0$) is uniformly convergent. Then we note that

$$s_2(x) := u(x+2) - u(x+1) \ge 0, \tag{2.1}$$

and

$$s_{2n}(x) := \sum_{k=1}^{2n}(-1)^n u(x+k) = s_{2n-2}(x) + (u(x+2n-1) - u(x+2n)) \ge s_{2n-2}(x) \ge 0, \tag{2.2}$$

and

$$\begin{aligned} s_{2n}(x) &= u(x+1) - [(u(x+2) - u(x+3)) + (u(x+4) - u(x+5)) \\ &+ \ldots + (u(x+2n-2) - u(x+2n-1)) + u(x+2n)] \le u(x+1) \le u(x). \end{aligned} \tag{2.3}$$

Therefore, the sequence $\{s_{2n}(x)\}$ of the even partial sums is increasing and bounded. Hence, the sequence is convergent and satisfies the inequality

$$\sum_{n=1}^{\infty}(-1)^{n-1}u(x+n) = \lim_{n\to\infty} s_{2n}(x) \le u(x) \qquad (\forall x > 0), \tag{2.4}$$

which shows that

$$\sum_{n=1}^{\infty}(-1)^{n-1}u(x+n)=O(u(x)) \qquad (x\to\infty). \tag{2.5}$$

One does not find a similar result about the asymptotic behavior of a function defined by the series $\sum_{n=1}^{\infty}u(x+n)$. For example, let us consider $u(x):=\exp(-x)\exp\left[-\int_0^x \exp(-t)dt\right]$

Then it is easily checked that *u(x)* is positive, monotonically decreasing and

$$\sum_{n=0}^{\infty}u(n+x)<\sum_{n=0}^{\infty}u(n)=\sum_{n=0}^{\infty}\exp(-n)\exp\left[-\int_0^n \exp(-t)dt\right]<\sum_{n=0}^{\infty}\exp(-n)=\frac{1}{1-\frac{1}{e}}.$$

However, we have

$$\sum_{n=0}^{\infty}u(n+x)\sim\int_x^{\infty}u(t)dt=\exp\left[-\int_0^x \exp(-t)dt\right],$$

that is

$$\sum_{n=0}^{\infty}u(n+x)\sim\exp(x)u(x) \text{ as } x\to\infty.$$

In view of the above example it seems very unlikely to have a general result about the asymptotic of such series. However, we have a restricted result: Let us define $f(x)\sim g(x)$ $(x\to\infty)$ if $\dfrac{f(x)}{g(x)}\to 1$ $(x\to\infty)$.

Let $u(x)$ $(x\ge x_0>0)$ *be a continuous function such that* $u(x)\sim\dfrac{1}{x^s}$ $(s=\sigma+i\tau$, $\sigma>0$, $x\to\infty$ *) and* $\sum_{n=1}^{\infty}u(x+n)$ $(x\ge x_0>0)$ *is uniformly convergent. Then*

$$\sum_{n=1}^{\infty}u(x+n)=O(x^{1-\sigma}) \qquad (x\to\infty). \tag{2.6}$$

Proof Assume that $0<\sigma\le 1$. Since the series is uniformly convergent this implies the existence of an integer N such that $\sup_x\{\sum_{n>N}u(x+n)\}<1$, *and* $\sup_x\{\sum_{n=1}^{N}u(x+n)\}<C$.

Therefore, the series $\sum_{n=1}^{\infty} u(x+n)$ *is bounded and trivially of the size* $O(x^{1-\sigma})$. *Secondly, assume that* $\sigma > 1$. *In this case, we have* $\left|\sum_{n=1}^{\infty} u(x+n)\right| \le C_1 \sum_{n=1}^{\infty} (x+n)^{-\sigma} \le C_2 x^{1-\sigma}$ *and we are done.* We give an example where the asymptotic relation (2.6) can be verified:

Consider $\sum_{n=1}^{\infty} u(x+n)$ ($x \ge x_0 > 0$), where

$$u(x) := x^{-s} \qquad (\sigma > 1). \tag{2.7}$$

In this particular case,

$$\sum_{n=1}^{\infty} u(x+n) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} e^{-xt} dt . \tag{2.8}$$

However,

$$\frac{1}{\Gamma(s)} \frac{t^{s-1}}{e^t - 1} \sim \frac{1}{\Gamma(s)} t^{s-2} \qquad (t \to 0^+). \tag{2.9}$$

Therefore, by Watson's lemma ([8], p. 5),

$$\sum_{n=1}^{\infty} u(x+n) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} e^{-xt} dt \sim \frac{\Gamma(s-1) x^{-s}}{\Gamma(s)} = \frac{x^{1-s}}{s-1} \qquad (x \to \infty), \tag{2.10}$$

what is in agreement with the (2.6).

3. The Main conjecture

Consider the function $u(x) := x^{-s}$ ($s := \sigma + i\tau$, $\sigma > 1$), let $\mu(n)$ denote the Möbius function ([4], p. 217), and let $(s)_0 := 1$, $(s)_k := \Gamma(s+k)/\Gamma(s)$ be the Pochhammer symbol.

Then

$$\sum_{n=1}^{\infty} \mu(n) u(x+n) = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n+x)^s} = \sum_{k=0}^{\infty} \frac{(s)_k (-x)^k}{k!} \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{k+s}}\right) = \sum_{k=0}^{\infty} \frac{(s)_k (-x)^k}{\zeta(s+k) k!}$$

$$(0 \le x < 1,\ \sigma > 1), \qquad (3.1)$$

which extends the well-known identity ([4], p. 260(1))

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \qquad (\sigma > 1). \qquad (3.2)$$

The RHS of (3.1) can be represented as an inverse Mellin transform ([4], p. 224):

$$\sum_{n=1}^{\infty} \mu(n) u(x+n) = \sum_{k=0}^{\infty} \frac{(s)_k (-x)^k}{\zeta(s+k) k!} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)\zeta(s-z)} x^{-z} dz$$

$$(\sigma > 1,\ 0 < c_1 \le c \le c_2 < \sigma - 1). \qquad (3.3)$$

Using the asymptotic relation ([1], p. 6, eq. (1.45))

$$\Gamma(s) = \Gamma(\sigma \pm i\tau) = \sqrt{2\pi}\, |\tau|^{\sigma-1/2} \exp(-\frac{\pi}{2}|\tau|)(1 + \mathrm{O}(1/|\tau|)$$

$$(|\tau| \to \infty, -\infty < a \le \sigma \le b < \infty), \qquad (3.4)$$

we have ($z := x + iy$ and fixed s),

$$|\Gamma(z)\Gamma(s-z)| \le C |y|^{\sigma-1} e^{-\pi|y|} \qquad (|y| \to \infty). \qquad (3.5)$$

In the integrand of (3.3), as we have $\mathrm{Re}(s-z) \ge \sigma_0 > 1$ and the fact that the zeta function does not vanish in the region $\sigma > 1$, the factor $1/\zeta(s-z)$ remains bounded in the region $0 < c_1 \le c \le c_2 < \sigma - 1$. Consequently, in view of the asymptotic representation (3.5) and due to the fact that the factor $1/\zeta(s-z)$ remains bounded in the region $0 < c_1 \le c \le c_2 < \sigma - 1$, the inverse Mellin transform integral (3.3) exists, and it can be evaluated by Cauchy's residue theorem. *The presence of the exponential term in (3.5) shows that the error term tends to zero faster than any power as* $y \to \infty$ ([9], p.148). The poles $z = -n$ ($n = 0, 1, 2, \ldots$) of the integrand in (3.3) are on the LHS of the line of integration. The integrand also has poles to the RHS of the line of integration at $z = s + n$ $(n = 0,1,2,...)$ and at $z = s - \rho$, where the $\rho's$ are the non-trivial zeros of the zeta function. Taking the sum over the residues gives, for large values of x ([9], p. 148),

$$\sum_{n=1}^{\infty} \mu(n)u(x+n) \sim \sum_{n=0}^{\infty} R_n x^{-s-n} + \sum_{\rho} \Re_{\rho} x^{\rho-s} \qquad (x \to \infty), \tag{3.6}$$

where

$$R_n := x^{s+n} \operatorname{Res}[\frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)\zeta(s-z)} x^{-z}; s+n], \tag{3.7}$$

$$\Re_{\rho} := x^{s-\rho} \operatorname{Res}[\frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)\zeta(s-z)} x^{-z}; s-\rho]. \tag{3.8}$$

It is to be remarked that *if all zeros of the zeta function are simple then* $\Re_{\rho}$ *does not depend on* x *and* hence the asymptotic behavior of (3.6) is governed by the power of x *with the largest positive real part*, that is,

$$\sum_{n=1}^{\infty} \mu(n)u(x+n) = O(x^{\sigma_M - \sigma}) \qquad (x \to \infty), \tag{3.9}$$

where $\sigma_M := \sup\{\operatorname{Re}\rho : \zeta(\rho) = 0\}$. It is to be remarked that if the bound $\sigma_M := \sup\{\operatorname{Re}\rho : \zeta(\rho) = 0\}$ is achieved, then (3.9) can be rewritten to give

$$\sum_{n=1}^{\infty} \mu(n)u(x+n) = O(x^{\sigma_M - \sigma)} \qquad (x \to \infty). \tag{3.10}$$

For the particular case $s = 3/2 + i\tau$, (3.9), which holds under the assumption of the simplicity of the zeros of the zeta function, becomes

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{(n+x)^{\frac{3}{2}+i\tau}} = O(x^{\sigma_M - \frac{3}{2}}) \qquad (x \to \infty) \tag{3.11}$$

At this point – on the analogy of (2.5) and (2.6) – we are tempted to make the following conjecture:

Conjecture *Let* $u(x)$ $(x \geq x_0 > 0)$ *be a continuous function such that* $u(x) \sim \frac{1}{x^s}$ $(s := \sigma + i\tau$, $\sigma > 1$, $x \to \infty)$ *and* $\sum_{n=1}^{\infty} u(x+n)$ $(x \geq x_0 > 0)$ *is uniformly convergent. Then*

$$\sum_{n=1}^{\infty} \mu(n) u(x+n) = O(x^{\frac{1}{2}-\sigma}) \qquad (x \to \infty), \tag{3.12}$$

where $\mu(n)$ *is the Möbius function.*

This conjecture would imply RH and resolve the simplicity of the zeros of the zeta function. It is because, for the particular case $s = 3/2 + i\tau$, we would have from (3.10)

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{(n+x)^{\frac{3}{2}+i\tau}} = O(\frac{1}{x}) \qquad (x \to \infty), \tag{3.13}$$

that is,

$$\sigma_M := \sup\{\operatorname{Re}\rho : \zeta(\rho) = 0\} = 1/2. \tag{3.14}$$

It is to be remarked that the particular case (3.13) would resolve the simplicity of the zeros of the zeta function as well.

4. Discussion and concluding remarks

There are well over one hundred known equivalents or consequences of RH. (Recent reviews [10, 11] list some fifty, covering a span of a century, from Helge von Koch's classic 1901 paper [12] on the connection between RH and the *Prime Number Theorem* to Baez-Duarte's new versions of the *Nyman-Beurling Criterion* in 2002 [13].) Our conjecture is stronger than RH; its particular case (for $\sigma = \frac{3}{2}$) is equivalent to RH. Because of the simple formulation and analytic tractability of this conjecture, we envisage that it would serve as a launching pad for many new attacks on RH. A less drastic conjecture may be stated under the conditions of the conjecture as:

$$\sum_{n=1}^{\infty} \mu(n) u(x+n) = O(x^{\frac{1}{2}-\sigma+\varepsilon}) \qquad (x \to \infty,\ \forall \varepsilon > 0). \tag{4.1}$$

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