Department of Mathematical Sciences
KFUPM
Term 061

MATH 202-08/ Quiz#3/ Time allowed=30 minutes

Q 1. [10 points] Consider the differential equation :

$$y'' + y = x\cos(x) + \sin(x). \tag{1}$$

a) Solve the homogeneous differential equation $y'' + y = 0$.

b) Give an optimal annihilator L for $x\cos(x) + \sin(x)$.

c) Apply the operator L to (1) and write the obtained differential equation. Then solve it.

d) Find a particular solution y_p to (1).

e) Write the general solution of (1).

<u>Solution:</u>

a) To solve the homogeneous equation

$$y'' + y = 0 \tag{2}$$

we write its auxiliary equation $m^2 + 1 = 0$ which has the roots $m_1 = -i$ and $m_2 = i$. Therefore we obtain the general solution :

$$y = c_1\cos(x) + c_2\sin(x),$$

where c_1 and c_2 are arbitrary real constants.

b) Since D^2+1 is an annihilator of $\sin(x)$ and $(D^2+1)^2$ is an annihilator of $x\cos(x)$, we deduce that $L = (D^2+1)^2$ is an optimal annihilator of $x\cos(x) + \sin(x)$.

c) Applying the operator L to both sides of (1), we get the equation

$$(D^2+1)^3 y = 0. \tag{3}$$

The auxiliary equation of (3) is given by

$$(m^2+1)^3 = 0$$

which has the roots $m_1 = m_2 = m_3 = -i$, $m_4 = m_5 = m_6 = i$. It follows that the general solution of (2) is given by

$$y = c_1\cos(x) + c_2\sin(x) + c_3x\cos(x) + c_4x\sin(x) + c_5x^2\cos(x) + c_6x^2\sin(x). \tag{4}$$

d) Since $c_1\cos(x)+c_2\sin(x)$ is the general solution of the homogeneous differential equation (2), we deduce that this part will not contribute to a particular solution of (1). Therefore a particular solution of (1) will be found in the form

$$y_p = c_3 x\cos(x) + c_4 x\sin(x) + c_5 x^2\cos(x) + c_6 x^2\sin(x)$$

or

$$y_p = Ax\cos(x) + Bx\sin(x) + Cx^2\cos(x) + Dx^2\sin(x). \tag{5}$$

Note that we have

$$\begin{aligned} y_p' &= A\cos(x) - Ax\sin(x) + B\sin(x) + Bx\cos(x) \\ &\quad +2Cx\cos(x) - Cx^2\sin(x) + 2Dx\sin(x) + Dx^2\cos(x), \end{aligned}$$

$$\begin{aligned} y_p'' &= 2(D-A)\sin(x) + 2(B+C)\cos(x) - (B+4C)x\sin(x) \\ &\quad +(4D-A)x\cos(x) - Cx^2\cos(x) - Dx^2\sin(x). \end{aligned} \tag{6}$$

Substituting (5) and (6) in (1), we get for all x

$$2(D-A)\sin(x) + 2(B+C)\cos(x) - 4Cx\sin(x) + 4Dx\cos(x) = x\cos(x) + \sin(x). \tag{7}$$

We deduce from (7) that

$$\begin{cases} 2(D-A) = 1 \\ 2(B+C) = 0 \\ -4C = 0 \\ 4D = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = -1/4 \\ B = 0 \\ C = 0 \\ D = 1/4 \end{cases}$$

Hence

$$y_p = -\frac{1}{4}x\cos(x) + \frac{1}{4}x^2\sin(x).$$

e) We conclude from a) and d) that the general solution of (1) is

$$y = c_1\cos(x) + c_2\sin(x) - \frac{1}{4}x\cos(x) + \frac{1}{4}x^2\sin(x),$$

where c_1 and c_2 are arbitrary real constants.