Department of Mathematical Sciences
KFUPM
Term 061

MATH 202-08/ Quiz#2/ Time allowed=25 minutes

Q 1. [4 points] Show that the functions $y_1 = \cosh(3x)$ and $y_2 = \sinh(3x)$ form a fundamental set of solutions of the differential equation $y'' - 9y = 0$ on the interval $(-\infty, +\infty)$.

Solution:

• y_1 and y_2 are solutions of the DE $y'' - 9y = 0$:

We have $y_1' = 3\sinh(3x)$ and $y_1'' = 9\cosh(3x)$. Therefore
$y_1'' - 9y_1 = 9\cosh(3x) - 9\cosh(3x) = 0$.

In the same way we have $y_2' = 3\cosh(3x)$ and $y_2'' = 9\sinh(3x)$. Therefore
$y_2'' - 9y_2 = 9\sinh(3x) - 9\sinh(3x) = 0$.

• y_1 and y_2 are linearly independent:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cosh(3x) & \sinh(3x) \\ 3\sinh(3x) & 3\cosh(3x) \end{vmatrix} \\ &= 3\cosh^2(3x) - 3\sinh^2(3x) = 3(\cosh^2(3x) - \sinh^2(3x)) \\ &= 3(1) = 3 \neq 0. \end{aligned}$$

We conclude that y_1 and y_2 form a fundamental set of solutions of the differential equation $y'' - 9y = 0$ on $(-\infty, +\infty)$. □

Q 2. [2 points] Find a particular solution of the differential equation
$y''' + 2y = x^2$ as a polynomial function.

Solution:

Since the righthand side is the polynomial function x^2 of degree 2 and the order of the equation is 3, it is clear that a polynomial function $y = c_2x^2$ is a candidate for a particular solution.

Since $(c_2x^2)''' = 0$, $y = c_2x^2$ is a solution of $y''' + 2y = x^2$ iff $2y = x^2$ or $2c_2x^2 = x^2$, which is equivalent to $2c_2 = 1$ or $c_2 = 1/2$.

Hence we get the particular solution $y = \frac{1}{2}x^2$. □

Q 3. [4 points] Given that $y_1 = \cos(2x)$ is a solution of the differential equation $y'' + 4y = 0$, give a second solution of this differential equation by using the method of reduction of order.

<u>Solution</u>:
We look for a second solution in the form $y = uy_1$. We have:

- $y' = uy_1' + u'y_1$
- $y'' = uy_1'' + 2u'y_1' + u''y_1$
- $y''+4y = uy_1''+2u'y_1'+u''y_1+4uy_1 = u(y_1''+4y_1)+2u'y_1'+u''y_1 = 2u'y_1'+u''y_1$ $=y_1(u'' + 2\frac{y_1'}{y_1}u')$.

Then $y = uy_1$ is a solution of $y'' + 4y = 0$ on $(-\infty, +\infty)$ iff

$$u'' + 2\frac{y_1'}{y_1}u' = 0 \quad \text{or} \quad v' - 4\tan(2x)v = 0 \quad \text{with} \quad v = u'. \tag{1}$$

An integrating factor of (1) is given by

$$e^{\int -4\tan(2x)dx} = e^{2ln(|\cos(2x)|)} = e^{ln(\cos^2(2x))} = \cos^2(2x).$$

Multiplying (1) by $\cos^2(2x)$, we get:

$$v'\cos^2(2x) - 4v\cos(2x)\sin(2x) = 0 \Rightarrow (v\cos^2(2x))' = 0$$
$$\Rightarrow v\cos^2(2x) = c \Rightarrow v = \frac{c}{\cos^2(2x)}, \quad \text{for some constant } c.$$

Taking c=2, we obtain $v = \frac{2}{\cos^2(2x)}$ or $u' = \frac{2}{\cos^2(2x)}$. We deduce that

$$u = \int \frac{2dx}{\cos^2(2x)} = \tan(2x).$$

It follows that $y_2 = \tan(2x)\cos(2x) = \sin(2x)$ is a second solution of the DE $y'' + 4y = 0$. $\square$