A New Formulation of the Dam Problem

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Abstract

We study a heterogeneous dam supplied by two reservoirs, for which we propose a new formulation based on the stream function. Without any monotonicity assumption on the permeability matrix, we prove that the free boundary is a continuous curve of the form $x = \phi(y)$. We also prove the uniqueness of the solution.

Introduction

The dam problem consists of studying the filtration of a fluid through a porous medium Ω subject to Darcy's law and to some boundary conditions. We look for the saturated region S (see Figure 1) and the fluid pressure p inside Ω. The interface $\Gamma = \partial[p > 0] \cap \Omega$ which separates the wet and dry regions is called a free boundary.
This problem has been studied by many authors. First Baiocchi, [5], solved the case of a homogeneous rectangular dam supplied by two reservoirs of water, by using the theory of variational inequalities. He also proved that Γ is an analytic curve of the form $y = \Phi(x)$. Benci extended Baiocchi's results to rectangular heterogeneous dams with permeability of the form $k_1(x)k_2(y)I_2$, where I_2 is the 2-by-2 unit matrix. In particular he proved (see [6]) that Γ is a continuous decreasing curve $y = \Phi(x)$ if $k_2'(y) \geq 0$. In [8], Caffarelli and Friedman proved that Γ is a curve of the form $x = \Psi(y)$ if $k_1(x) = 1$ and $k_2(y)$ is a non-increasing step function.

Unfortunately the variational inequalities approach is not possible for dams with general geometry. Later on Alt in [1] and Brezis, Kinderlehrer, and Stampacchia in [7] introduced a new approach. Moreover, for the homogeneous dam Alt proved (see [2]) that the free boundary is an analytic curve $y = \Phi(x)$. Uniqueness of the S_3-connected solution was proved by Carrillo and Chipot in [9] and also by Alt and Gilardi in [4].

In [12] and [15], the authors showed that if the permeability matrix is given by $a(X) = k(x, y)I_2$ with $\partial k/\partial y \geq 0$ in $\mathcal{D}'(\Omega)$, then Γ is a continuous curve $y = \Phi(x)$ and the S_3-connected solution is unique. These results were generalized by the second author in [14] to the case where the permeability matrix $a(X) = (a_{ij}(X))$ is such that $a_{12}(X) \equiv 0$ and $\partial a_{22}/\partial y \geq 0$ in $\mathcal{D}'(\Omega)$. Recently the authors have considered the case where $a_{12}, a_{22} \in C^1$ and $div(a_{12}, a_{22}) \geq 0$. They proved, via a change of variables, that Γ is locally represented by continuous curves.

In this paper, we consider a heterogeneous dam supplied by two reservoirs each containing the same fluid. Our main objective is to study the free boundary, and to establish some of its properties without assuming any kind of monotonicity on the permeability. This is done by introducing a new formulation using the stream function instead of the classical one based on the pressure. In this formulation the total flux through the left-hand side of the dam is prescribed instead of prescribing the height of the left reservoir and the pressure on the contact zone between the reservoir and the dam. Note that a similar formulation has been introduced in [3] to study a homogenous dam supplied by two reservoirs each containing two immiscible fluids.
We first prove a monotonicity result for the function characterizing the wet region. This allows us to define the free boundary as a curve of the form $x = \phi(y)$. Then we prove the continuity of ϕ and the uniqueness of the solution.

1 Statement of the Problem

Throughout this paper, we will denote by X the point of coordinates (x, y). Sometimes we will also use the notation (x_P, y_P) to denote the coordinates of the point P. We will denote by $\chi(E)$ the characteristic function of the set E.

We consider a dam which is a porous medium represented by a bounded domain Ω of $\mathbb{R}^2$ with boundary $\partial\Omega = \overset{\frown}{AB} \cup \overset{\frown}{BT} \cup \overset{\frown}{TA}$. We assume that

$$\begin{aligned}
&\overset{\frown}{TA} = \{(a(y), y),\ y \in [0, y_T]\} && \text{with } y \mapsto a(y) \text{ an increasing } C^1 \text{ function}\\
&\overset{\frown}{BT} = \{(b(y), y),\ y \in [y_B, y_T]\} && \text{with } y \mapsto b(y) \text{ a decreasing } C^1 \text{ function}\\
&\overset{\frown}{AB} = \{(x, c(x)),\ x \in [x_A, x_B]\} && \text{with } x \mapsto c(x) \text{ a decreasing } C^1 \text{ function.}
\end{aligned}$$

The dam is supplied by two reservoirs of water (see Figure 1). To the left, we assume that the total entering flux through $\overset{\frown}{A_1 A}$ is equal to the constant $Q > 0$

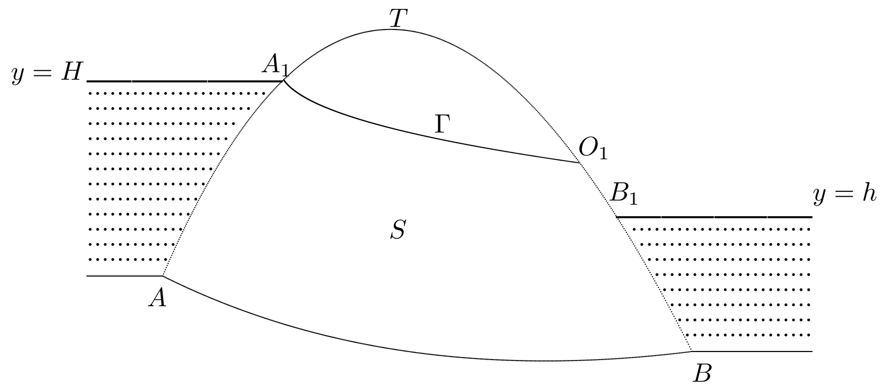

Figure 1

and the fluid in the reservoir is at the level $H(Q) = y_{A_1}$ depending on Q, while to the right it is at the level $h = y_{B_1}$ independent on Q, with $H > h > 0$.
We assume that the flow in Ω has reached a steady state and we denote by S the saturated region of the porous medium. The boundary ∂S of S is divided into four parts (see Figure 1) : the impervious part $\widehat{AB}$, the free boundary Γ, $\widehat{BB_1}$ and $\widehat{A_1A}$ representing the part in contact with the reservoirs and, finally, $\widehat{B_1O_1}$, the part where the fluid flows out of Ω and not directly into a reservoir.
The flow is governed by the following Darcy law

$$v = -a(X)\nabla(p + y) \qquad (1.1)$$

where v is the fluid velocity, p is its pressure, and $a : \bar{\Omega} \longrightarrow I\!R^{2\times 2}$ is the permeability matrix satisfying

$$\begin{cases} i) & a \in C^{0,\alpha}(\Omega), \qquad \max_{i,j} |a_{i,j}|_{0,\Omega} \leq M \\ ii) & a(X)\xi.\xi \geq m|\xi|^2 \qquad \text{for all } \xi \in I\!R^2, \quad X \in \Omega, \end{cases} \qquad (1.2)$$

for some $m, M > 0$ and $\alpha \in (0, 1)$ and where $|.|_{0,\Omega}$ is the L^∞ norm.

Since $\widehat{AB}$ is assumed to be impervious, we have

$$v.\nu = 0 \quad \text{on} \quad \widehat{AB}, \qquad (1.3)$$

where ν is the exterior unit normal vector to ∂S.
We assume that there is no flux through the free boundary and that p agrees with the atmospheric pressure normalized at zero. So we have

$$p = 0 \quad \text{and} \quad v.\nu = 0 \quad \text{on} \quad \Gamma. \qquad (1.4)$$

On the part of $\partial\Omega$ in contact with the reservoirs, we assume that the pressure is given by the exterior fluid pressure, i.e. we have

$$p = h - y \quad \text{on } \widehat{BB_1}, \qquad p = H - y \quad \text{on } \widehat{A_1A}\ . \tag{1.5}$$

Finally on $\widehat{B_1O_1}$, p coincides with the atmospheric pressure and the water is free to exit the porous medium. Thus we have

$$p = 0 \quad \text{and} \quad v.\nu \geq 0 \quad \text{on } \widehat{B_1O_1}\ . \tag{1.6}$$

We assume the flow to be incompressible, i.e. $div(v) = 0$ in S. Then there exists a stream function ψ such that

$$v = -Rot\psi = -\Big(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\Big) = -a(X)\nabla(p+y) \quad \text{in } S. \tag{1.7}$$

Let $\zeta \in \mathcal{D}(S)$. We have by (1.1) and (1.7)

$$0 = \int_S \nabla(p+y).Rot\zeta = \int_S a^{-1}(X)Rot\psi.Rot\zeta = \int_S b(X)\nabla\psi.\nabla\zeta \tag{1.8}$$

where $b(X) = \dfrac{{}^t a(X)}{det(a(X))}$, $det(a(X))$ is the determinant of the matrix $a(X)$ and ${}^t a(X)$ is its transpose. Note that we deduce from (1.2) the existence of $\lambda, \Lambda > 0$ such that

$$\begin{cases} i) \quad b \in C^{0,\alpha}(\Omega), \qquad \max_{i,j} |b_{i,j}|_{0,\Omega} \leq \Lambda \\ ii) \quad b(X)\xi.\xi \geq \lambda|\xi|^2 \qquad \text{for all } \xi \in I\!R^2, \quad X \in \Omega. \end{cases} \tag{1.9}$$

We obtain from (1.8)

$$div(b(X)\nabla\psi) = 0 \qquad \text{in } \mathcal{D}'(S). \tag{1.10}$$

Let us set $\nu = (\nu_x, \nu_y)$ and denote the tangent vector by $\tau = (-\nu_y, \nu_x)$. Then

$$v.\nu = -Rot\psi.\nu = \frac{\partial\psi}{\partial\tau}. \tag{1.11}$$

Moreover one can verify that

$$b(X)\nabla\psi.\nu = \frac{\partial(p+y)}{\partial\tau}. \tag{1.12}$$

It follows from (1.3) and (1.11) that ψ is constant along $\widehat{AB}$. So there exists a constant c_1 such that

$$\psi = c_1 \quad \text{on } \widehat{AB}\ . \tag{1.13}$$

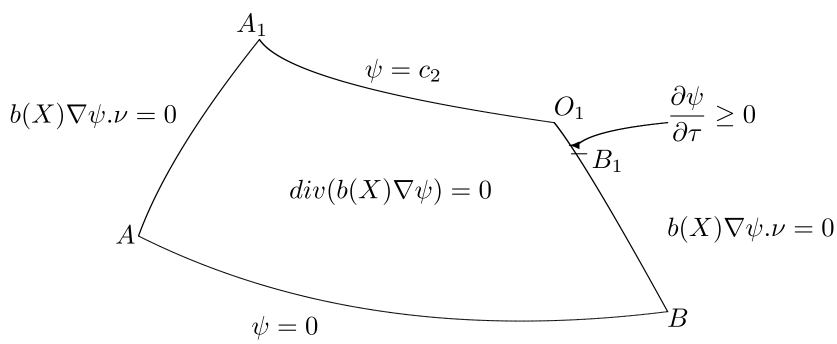

Figure 2

In the same way ψ is constant along Γ. So there exists a constant c_2 such that

$$\psi = c_2 \quad \text{on } \Gamma. \tag{1.14}$$

Now since $p = 0$ on Γ, we obtain from (1.12)

$$b(X)\nabla\psi.\nu = \nu_x \quad \text{on } \Gamma. \tag{1.15}$$

By (1.5), $p + y$ is constant along $\widehat{A_1A}$ and $\widehat{BB_1}$. It follows then by (1.12)

$$b(X)\nabla\psi.\nu = 0 \quad \text{on } \widehat{A_1A} \cup \widehat{BB_1}\ . \tag{1.16}$$

Finally we deduce from (1.6) and (1.11)-(1.12) that

$$b(X)\nabla\psi.\nu = \nu_x \quad \text{and} \quad \frac{\partial\psi}{\partial\tau} \geq 0 \quad \text{on } \widehat{B_1O_1}\ . \tag{1.17}$$

We choose $c_1 = 0$ and we are going to prove, in a formal way, that $c_2 > 0$. First we claim that $c_2 \geq 0$. Indeed, suppose that $c_2 < 0$ (see Figure 2).
To begin with, since $h < H$, we expect that the point O_1 is located above B_1, and in particular $\widehat{O_1B_1} \neq \emptyset$. Next, since by (1.17) ψ increases from B_1 to O_1, it takes its minimum value on $\widehat{A_1A}$ or on $\widehat{BB_1}$. Suppose there exists $m_0 \in \widehat{BB_1}$ at which ψ achieves its minimum value. Then $\frac{\partial\psi}{\partial\tau}(m_0) = 0$, and consequently, we have $0 = b(m_0)\nabla\psi.\nu(m_0) = \frac{\partial\psi}{\partial\nu}(m_0)b(m_0)\nu.\nu$. From (1.9)$ii$) we deduce that $\frac{\partial\psi}{\partial\nu}(m_0) = 0$, which contradicts the Hopf maximum principle.
Similarly, we get a contradiction if ψ achieves its minimum on $\widehat{A_1A}$.
Now, we assert that $c_2 > 0$. Indeed if $c_2 = 0$, then we cannot have as before, a minimum neither on $\widehat{A_1A}$ nor on $\widehat{BB_1}$. So the minimum is achieved on $\widehat{O_1A_1} \cup \widehat{AB}$

$\cup\, B_1\overset{\frown}{O_1}$. But since $\dfrac{\partial\psi}{\partial\tau} \geq 0$ on $B_1\overset{\frown}{O_1}$ and $\psi = 0$ on $O_1\overset{\frown}{A_1} \cup \overset{\frown}{AB}$, the minimum will be achieved at a point $m_0 \in B_1\overset{\frown}{O_1}$. We necessarily have, as before, $\dfrac{\partial\psi}{\partial\tau}(m_0) = 0$, which leads to $(b(m_0)\nu.\nu)\dfrac{\partial\psi}{\partial\nu}(m_0) = \nu_x > 0$. Hence by (1.9)$ii)$ $\dfrac{\partial\psi}{\partial\nu}(m_0) > 0$ and we get a contradiction to the Hopf maximum principle.

Remark 1.1. *i) We have $c_2 = Q$ is the total flux through $A_1\overset{\frown}{A}$. Indeed we have by (1.11)*

$$Q = \int_A^{A_1} -v.\nu = \int_A^{A_1} -\frac{\partial\psi}{\partial\tau} = \psi(A_1) - \psi(A) = c_2 - 0 = c_2.$$

Moreover by the strong maximum principle we have $0 < \psi < Q$ in S.

ii) Since $b(X)\nabla\psi.\nu = \nu_x$ and $\psi = Q$ on Γ, one has $\dfrac{\partial\psi}{\partial\nu}b(X)\nu.\nu = \nu_x$ on Γ. Moreover by the maximum principle and the positivity of $b(X)$, we deduce that $\nu_x \geq 0$ on Γ. This means that Γ is a curve $x = \phi(y)$. This will be justified more rigourously in section 2.

In order to derive a weak formulation for our problem, we first extend ψ by Q to $\Omega \setminus S$. Next we choose a test function $\zeta \in H^1(\Omega)$ with $\zeta = 0$ on $\overset{\frown}{AB}$. Taking into account (1.10), (1.15)-(1.17), we obtain, in a formal way,

$$\int_\Omega b(X)\nabla\psi.\nabla\zeta = \int_S b(X)\nabla\psi.\nabla\zeta = \int_{\partial S} b(X)\nabla\psi.\nu\zeta = \int_{\Gamma\cup B_1\overset{\frown}{O_1}} \nu_x\zeta. \qquad (1.18)$$

Assuming Γ smooth enough, we have

$$\int_\Omega \chi([\psi = Q])\zeta_x = \int_{\partial\Omega} \zeta\nu_x - \int_{\partial S} \zeta\nu_x = \int_{O_1\overset{\frown}{T}\cup T\overset{\frown}{A_1}} \nu_x\zeta - \int_\Gamma \nu_x\zeta$$

which leads to

$$\int_\Gamma \nu_x\zeta = -\int_\Omega \chi([\psi = Q])\zeta_x + \int_{O_1\overset{\frown}{T}\cup T\overset{\frown}{A_1}} \nu_x\zeta. \qquad (1.19)$$

Comparing (1.18) and (1.19), we get

$$\int_\Omega b(X)\nabla\psi.\nabla\zeta + \int_\Omega \chi([\psi = Q])\zeta_x = \int_{T\overset{\frown}{A_1}} \nu_x\zeta + \int_{B_1\overset{\frown}{T}} \nu_x\zeta. \qquad (1.20)$$

Since we are assuming that $\psi = Q$ above the level $y = H$, we can rewrite (1.20) as

$$\int_\Omega \big(b(X)\nabla\psi + \chi([\psi = Q])e_x\big).\nabla\zeta - \int_{\overset{\frown}{TA}} \chi([\psi = Q])\nu_x\zeta = \int_{B_1\overset{\frown}{T}} \nu_x\zeta.$$

where $e_x = (0,1)$. Hence we are led to the following weak formulation

$$(P)\begin{cases} \text{Find } (\psi,\gamma,\widetilde{\gamma}) \in H^1(\Omega)\times L^\infty(\Omega)\times L^\infty(\widehat{TA}) \text{ such that }: \\ (i) \quad \displaystyle\int_\Omega \big(b(X)\nabla\psi + \gamma e_x\big).\nabla\zeta \; - \int_{\widehat{TA}} \widetilde{\gamma}\nu_x\zeta \; = \int_{\widehat{B_1T}} \nu_x\zeta \\ \qquad\qquad\qquad \forall\zeta \in H^1(\Omega), \quad \zeta = 0 \text{ on } \widehat{AB} \\ (ii) \quad \gamma \in H(\psi - Q) \quad \text{a.e. in } \Omega, \qquad \widetilde{\gamma} \in H(\psi - Q) \quad \text{a.e. in } \widehat{TA} \\ (iii) \quad \psi = 0 \quad \text{on } \widehat{AB}, \end{cases}$$

where H is the heaviside graph.

Remark 1.2. *As mention in [3] in the case of two fluids, the new formulation is given in terms of prescribed flux Q instead of prescribed height of the left reservoir. Indeed in the formulation (P) one is given Q as well as B_1, but not A_1.*

The existence of a solution of (P) can be obtained by arguing as in [1] or [7]. In section 2, we give some properties of the solutions. In particular we prove the continuity of ψ. Then we show a monotonicity result of γ with respect to x. This allows us to define the free boundary as a curve $x = \phi(y)$. We also prove that the dam is wet below the line $y = h$. In section 3, we establish the continuity of ϕ and obtain the expression of γ as a characteristic function of the dry part of the dam. Finally, in section 4, we prove the uniqueness of the solution.

2 Some Properties of the Solutions

Throughout this section we denote a solution of (P) by $(\psi,\gamma,\widetilde{\gamma})$. First we have

Proposition 2.1.

$$div\big(b(X)\nabla\psi\big) = -\gamma_x \qquad in \quad \mathcal{D}'(\Omega). \tag{2.1}$$

Proof. It suffices to take $\zeta \in \mathcal{D}(\Omega)$ as a test function in $(P)i$). □

Proposition 2.2.

$$0 \le \psi \le Q \qquad in \quad \Omega. \tag{2.2}$$

Proof. i) Using ψ^- as a test function in $(P)i$), we obtain since $[\psi \le 0] \subset [\psi < Q]$, $\gamma = 0$ a.e. in $[\psi < Q]$ and $\widetilde{\gamma} = 0$ a.e. in $\widehat{TA} \cap [\psi < Q]$,

$$\int_\Omega b(X)\nabla\psi^-.\nabla\psi^- = -\int_{\widehat{B_1T}} \psi^-\nu_x \le 0 \qquad \text{since } \nu_x \ge 0 \text{ on } \widehat{B_1T}\ .$$

By (1.9)ii), we deduce that $\nabla\psi^- = 0$ a.e. in Ω. Then ψ^- is constant in Ω. Since $\psi = 0$ on $\widehat{AB}$, we obtain $\psi \geq 0$ in Ω.
ii) Taking $(\psi - Q)^+$ as a test function in $(P)i$), we obtain since $\gamma = 1$ a.e. in $[\psi > Q]$, $\widetilde{\gamma} = 1$ a.e. in $\widehat{TA} \cap [\psi > Q]$, and $\nu_x \geq 0$ on $\widehat{BB_1}$

$$\int_\Omega b(X)\nabla(\psi - Q)^+.\nabla(\psi - Q)^+ = -\int_\Omega (\psi - Q)^+_x + \int_{\widehat{TA}} (\psi - Q)^+\nu_x$$
$$+ \int_{\widehat{B_1T}} (\psi - Q)^+\nu_x = -\int_{\widehat{BB_1}} (\psi - Q)^+\nu_x \leq 0.$$

It follows that $(\psi - Q)^+$ is constant in Ω. Since $\psi = 0$ on $\widehat{AB}$, we obtain $\psi \leq Q$ in Ω. $\square$

Proposition 2.3.

$$\psi \in C^{0,\alpha}_{loc}(\overline{\Omega} \setminus \{A, B, T\}) \qquad \textit{for some} \quad \alpha \in (0,1).$$

Proof. First, since $\gamma \in L^\infty(\Omega)$, and $b \in L^\infty(\Omega)$ and is strictly elliptic, we deduce from (2.1) and the regularity theory for elliptic equations (see [13], Theorem 8.22, p. 200 and Theorem 8.29, p. 205) that $\psi \in C^{0,\alpha}_{loc}(\Omega \cup Int(\widehat{AB}))$ for some $\alpha \in (0,1)$.

For the regularity on $Int(\widehat{BT}) \cup Int(\widehat{TA})$, we use the reflection method. We will give the proof only for $Int(\widehat{TA})$, as it is the same for $Int(\widehat{BT})$.
Let $X_0 = (x_0, y_0) = (a(y_0), y_0) \in Int(\widehat{TA})$, $\delta > 0$ and consider the set

$$Z = \{(x, y) \,/\, a(y) - \delta < x < a(y) + \delta \quad \text{and} \quad y_0 - \delta < y < y_0 + \delta\}.$$

We assume that $\delta > 0$ is small enough so that $\{(a(y), y) \,/\, y_0 - \delta < y < y_0 + \delta\} \subset \widehat{TA}$ and (see Figure 3)

$$Z \cap \Omega = \{(x, y) \,/\, a(y) < x < a(y) + \delta \quad \text{and} \quad y_0 - \delta < y < y_0 + \delta\}.$$

Then we define the following functions in Z :

$$\overline{\psi}(x,y) = \begin{cases} \psi(x,y) & \text{if } (x,y) \in Z \cap \Omega \\ \psi(2a(y) - x, y) & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{b}(x,y) = \begin{cases} b(x,y) & \text{if } (x,y) \in Z \cap \Omega \\ c(2a(y) - x, y) & \text{otherwise;} \end{cases}$$

where

$$\begin{cases} c_{11} = b_{11} + 4a'^2(y)b_{22} - 2a'(y)(b_{12} + b_{21}) \\ c_{12} = 2a'(y)b_{22} - b_{12}, \; c_{21} = 2a'(y)b_{22} - b_{21}, \; c_{22} = b_{22}. \end{cases}$$

Let $f = \overline{\gamma}\chi(Z \setminus \Omega) + (2\widetilde{\gamma}(y) - \gamma)\chi(Z \cap \Omega)$, where $\overline{\gamma}(x,y) = \gamma(2a(y) - x, y)$ if $(x,y) \in Z \setminus \Omega$. Then we have

$$div(\overline{b}(x,y)\nabla\overline{\psi}) = f_x \quad \text{in } \mathcal{D}'(Z). \tag{2.3}$$

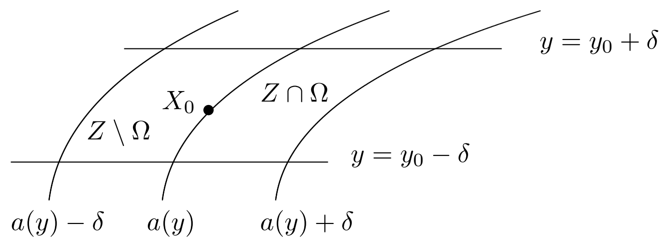

Figure 3

Indeed we have, for $\zeta \in \mathcal{D}(Z)$ and $\overline{\zeta}(x, y) = \zeta(2a(y) - x, y)$,

$$\begin{aligned} \int_{Z\setminus\Omega} c(2a(y) - x, y)\nabla\overline{\psi}.\nabla\zeta &= \int_{Z\cap\Omega} c(x, y)\nabla\overline{\psi}(2a(y) - x, y)\nabla\zeta(2a(y) - x, y) \\ &= \int_{Z\cap\Omega} b(x, y)\nabla\psi(x, y).\nabla\overline{\zeta}(x, y). \end{aligned} \tag{2.4}$$

Using (2.4) and the fact that $\zeta + \overline{\zeta}$ is a test function for (P), we get

$$\begin{aligned} \int_Z \overline{b}(x, y)\nabla\overline{\psi}.\nabla\zeta &= \int_{Z\cap\Omega} b(x, y)\nabla\psi.\nabla(\zeta + \overline{\zeta}) \\ &= -\int_{Z\cap\Omega} \gamma(\zeta + \overline{\zeta})_x + \int_{Z\cap\widehat{TA}} \widetilde{\gamma}(y)(\zeta + \overline{\zeta})\nu_x \\ &= \int_{Z\setminus\Omega} \overline{\gamma}\zeta_x + \int_{Z\cap\Omega} (2\widetilde{\gamma}(y) - \gamma)\zeta_x = \int_Z f\zeta_x. \end{aligned}$$

Now it is clear that the coefficients of $\overline{b}(x, y)$ and f are uniformly bounded in Z. Moreover we claim that $\overline{b}(x, y)$ is strictly elliptic in Z. Indeed it is enough to verify that $c(x, y)$ is strictly elliptic in $Z \cap \Omega$. Let $c^*(x, y)$ be the matrix defined by $c^*_{ii} = c_{ii}$ and $c^*_{ij} = -c_{ij}$ for $i \neq j$. Then we have for each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$

$$c(x, y).\xi.\xi = c_{11}\xi_1^2 + (c_{12} + c_{21})\xi_1\xi_2 + c_{22}\xi_2^2 = c^*(x, y).\xi^*.\xi^*, \quad \text{with } \xi^* = (\xi_1, -\xi_2).$$

Since $|\xi| = |\xi^*|$, it is clear that it is enough to show the strict ellipticity of $c^*(x, y)$. For this purpose we remark that $c^*(x, y) = {}^t d(x, y)b(x, y)d(x, y)$, where $d(x, y) = (d_{ij}(x, y))$ is the matrix defined by $d_{11}(x, y) = d_{22}(x, y) = 1$, $d_{12}(x, y) = 0$, and $d_{21}(x, y) = -2a'(y)$. Then we have by (1.9)$ii)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$

$$c^*(x, y)\xi.\xi = {}^t d(x, y)b(x, y)d(x, y)\xi.\xi = b(x, y)(d(x, y)\xi).(d(x, y)\xi) \geq \lambda|d(x, y)\xi|^2.$$

Since $|d(x,y)\xi|^2 = (1+4a'^2(y))\xi_1^2 - 4a'(y)\xi_1\xi_2 + \xi_2^2$, it is enough to choose η such that $0 < \eta < \min\Big(\frac{1}{2}, \frac{1}{16 \max\limits_{y_0-\delta\le y\le y_0+\delta} a'^2(y)}\Big)$, to get $|d(x,y)\xi|^2 \ge \eta|\xi|^2$.

Now using (2.3) and taking into account the fact that $f \in L^\infty(Z)$, $\overline{b} \in L^\infty(Z)$, and the strict ellipticity of $\overline{b}(x,y)$, we conclude (see [13], Theorem 8.22, p. 200) that $\overline{\psi} \in C^{0,\alpha}_{loc}(Z)$ for some $\alpha \in (0,1)$. Thus $\psi \in C^{0,\alpha}_{loc}(\Omega \cup Int(\widehat{TA}))$. □

The following monotonicity result plays a major role later.

Proposition 2.4.

$$\gamma_x \ge 0 \qquad in \quad \mathcal{D}'(\Omega). \tag{2.5}$$

Proof. Let $\zeta \in \mathcal{D}(\Omega)$, $\zeta \ge 0$. For $\epsilon > 0$, $\xi = \min\Big(\frac{Q-\psi}{\epsilon}, \zeta\Big)$ is a test function for $(P)i)$. Using the fact that $\xi = 0$ on $[\psi = Q]$ and $\gamma = 0$ on $[\psi < Q]$, we get

$$\int_{\Omega\cap[Q-\psi\ge\epsilon\zeta]} b(X)\nabla(Q-\psi).\nabla\zeta = -\frac{1}{\epsilon}\int_{\Omega\cap[Q-\psi<\epsilon\zeta]} b(X)\nabla(Q-\psi).\nabla(Q-\psi) \le 0.$$

Letting $\epsilon \to 0$ and using (2.1), we get the desired result. □

As a consequence of Propositions 2.1, 2.2 and (2.4), we obtain

Proposition 2.5.

$$\psi > 0 \qquad in \quad \Omega. \tag{2.6}$$

Proof. Indeed, using (2.1) and (2.5), we deduce that $div\big(b(X)\nabla\psi\big) \le 0$. Since moreover $\psi \ge 0$ in Ω, we obtain by the strong maximum principle that either $\psi \equiv 0$ in Ω or $\psi > 0$ in Ω. If we have $\psi \equiv 0$ in Ω, then $\gamma = 0$ a.e. in Ω and $\widetilde{\gamma} = 0$ a.e. in $\widehat{TA}$. It follows from $(P)i)$

$$\int_{\widehat{B_1T}} \zeta\nu_x = 0 \qquad \forall\zeta \in H^1(\Omega), \quad \zeta = 0 \text{ on } \widehat{AB}$$

which is impossible. Hence $\psi > 0$ in Ω. □

As a consequence of the continuity of ψ and the monotonicity of γ, we deduce the following key Proposition

Proposition 2.6. *Let $(x_0, y_0) \in \Omega$. Then we have*

i) *If* $\psi(x_0,y_0) = Q$ *then* $\psi(x,y_0) = Q$ *for all* $x \ge x_0$.

ii) *If* $\psi(x_0,y_0) < Q$ *then there exists* $\epsilon > 0$ *such that*

$$\psi(x,y) < Q \qquad \forall (x,y) \in D_\epsilon = \Big(B_\epsilon(x_0,y_0) \cup \big((-\infty,x_0)\times(y_0-\epsilon,y_0+\epsilon)\big)\Big) \cap \Omega.$$

Proof. It suffices to prove *ii*). Let $(x_0, y_0) \in \Omega$ be such that $\psi(x_0, y_0) < Q$. By continuity, there exists $\epsilon > 0$ such that $\psi < Q$ in $B_\epsilon(x_0, y_0)$. Then $\gamma = 0$ a.e. in $B_\epsilon(x_0, y_0)$. Since $\gamma_x \geq 0$ in $\mathcal{D}'(\Omega)$ and $0 \leq \gamma \leq 1$, we obtain $\gamma = 0$ a.e. in D_ϵ. This leads by (2.1) to $div(b(X)\nabla\psi) = 0$ in D_ϵ. Finally, given that $\psi \leq Q$ in Ω and $\psi < Q$ in $B_\epsilon(x_0, y_0)$, we deduce by the strong maximum principle that $\psi < Q$ in D_ϵ. □

The following theorem shows that the dam is wet under the level $y = h$.

Theorem 2.1. *We have*

$$\gamma = 0 \quad a.e.\ in \quad \Omega \cap [y < h] = \Omega_0 \qquad and \qquad \psi < Q \quad in \quad \Omega_0.$$

Proof. Let $u = Q - \psi$ and $\chi = 1 - \gamma$. Then we have $\chi = 1$ a.e. in $[u > 0]$ and $\chi \in [0, 1]$ a.e. in $[u = 0]$. Moreover for each $\zeta \in H^1(\Omega_0)$, $\zeta = 0$ on $\partial\Omega_0 \setminus \widehat{BB_1}$, we have

$$\int_{\Omega_0} \big(b(X)\nabla u + \chi e_x\big)\nabla\zeta = \int_{\Omega_0} \big(- b(X)\nabla\psi + (1-\gamma)e_x\big)\nabla\zeta$$
$$= -\int_{\Omega_0} \big(b(X)\nabla\psi + \gamma e_x\big)\nabla\zeta + \int_{\Omega_0} \zeta_x = \int_{\widehat{BB_1}} \zeta\nu_x.$$

Let $x_0 \in (x_A, x_B)$ and $y_1, y_2 \in (y_B, h)$ such that $y_1 < y_2 < h$ and $Z = \big((x_0, +\infty) \times (y_1, y_2)\big) \cap \Omega \subset \Omega_0$. Let $\eta \in \mathcal{D}\big(y_1, y_2\big)$ and $\zeta = (x - x_0)^+\eta$. Then for $\epsilon > 0$ and $H_\epsilon(s) = \min\big(\frac{s^+}{\epsilon}, 1\big)$, we have $(H_\epsilon(u) - 1)\zeta \in H^1(\Omega_0)$ and $(H_\epsilon(u) - 1)\zeta = 0$ on $\partial\Omega_0 \setminus \widehat{BB_1}$. It follows that

$$\int_Z \big(b(X)\nabla u + \chi e_x\big)\nabla((H_\epsilon(u) - 1)\zeta) = \int_{\partial Z \cap \widehat{BB_1}} (H_\epsilon(u) - 1)\zeta\nu_x$$

which we can rewrite as

$$\begin{aligned} \int_Z (1-\chi)e_x.\nabla\zeta \quad &= \quad \int_Z (1 - H_\epsilon(u))b(X)\nabla u.\nabla\zeta - \int_Z H'_\epsilon(u)\zeta b(X)\nabla u.\nabla u \\ &+ \quad \int_Z e_x.\nabla\zeta - \int_Z \chi e_x.\nabla(H_\epsilon(u)\zeta) + \int_{\partial Z \cap \widehat{BB_1}} (H_\epsilon(u) - 1)\zeta\nu_x. \end{aligned}$$

Since $H'_\epsilon(t) \geq 0$, $b(X)$ is positive, and $\chi\nabla(H_\epsilon(u)\zeta) = \nabla(H_\epsilon(u)\zeta)$ a.e. in Z, we get

$$\int_Z (1-\chi)e_x.\nabla\zeta \leq \int_Z (1 - H_\epsilon(u))b(X)\nabla u.\nabla\zeta.$$

Letting $\epsilon \to 0$, we obtain $\displaystyle\int_Z (1-\chi)\eta(y) = 0$. So $\chi = 1$ a.e. in Z. This holds for all such sub-domains Z of Ω_0. Hence $\chi = 1$ a.e. in Ω_0 and $\gamma = 0$ a.e. in Ω_0.

Now, we deduce from (2.1) that $div(b(X)\nabla\psi) = 0$ in Ω_0. Moreover, since $\psi \leq Q$ in Ω_0, we obtain by the strong maximum principle that either $\psi < Q$ in Ω_0 or $\psi \equiv Q$ in Ω_0. The last situation cannot happen since $\psi \in C^0(\Omega_0 \cup Int\,\widehat{AB})$ and $\psi = 0$ on $\widehat{AB}$. □

We end this section with a series of properties of the set $[\psi = Q]$.

Proposition 2.7. *Let* $B_1^* = (a(h), h)$, $E \in Int(\widehat{TB_1^*})$ *and* $l_E = \Omega \cap [y = y_E]$.

$$\text{If} \quad \psi = Q \quad \text{on} \quad l_E, \quad \text{then} \quad \psi \equiv Q \quad \text{in} \quad \Omega_E = \Omega \cap [y > y_E].$$

Proof. Note that $\zeta = (Q-\psi)\chi(\Omega_E)$ is a test function for (P) since $\zeta \in H^1(\Omega)$ and $\zeta = 0$ on $\widehat{AB}$. Given that $\gamma = 0$ a.e. in $[\psi < Q]$ and $\widetilde{\gamma} = 0$ a.e. in $\widehat{TA} \cap [\psi < Q]$, we have

$$\int_{\Omega_E} b(X)\nabla(Q-\psi).\nabla(Q-\psi) = -\int_{\widehat{B_1T}} \zeta\nu_x \leq 0.$$

It follows that $Q - \psi$ is constant in Ω_E. But since $\psi = Q$ on l_E, we obtain $\psi = Q$ in Ω_E. □

Proposition 2.8. *Assume that* $E \in Int(\widehat{TA})$, $\psi(E) = Q$, *and that the interior sphere condition is satisfied at the point* E*. Then there exists a sequence* $E_n \in \Omega$ *such that* $E_n \longrightarrow E$ *and* $\psi(E_n) = Q$.

Proof. We argue by contradiction and assume that there exists $\epsilon > 0$ such that $\psi < Q$ in $B_{2\epsilon}(E) \cap \Omega$. Let $\Omega_\epsilon = B_\epsilon(E) \cap \Omega$. Then we obtain, by taking into account the fact that $\gamma = 0$ a.e. in Ω_ϵ and $\widetilde{\gamma}\nu_x \leq 0$ on $\widehat{TA}$, that

$$\int_{\Omega_\epsilon} b(X)\nabla\psi.\nabla\zeta = \int_{\widehat{TA}} \widetilde{\gamma}\xi\nu_x \leq 0 \quad \forall\zeta \in H^1(B_\epsilon(E)),\ \zeta \geq 0,\ \zeta = 0 \quad \text{on } \partial\Omega_\epsilon \cap \Omega. \tag{2.7}$$

Now we consider v defined by

$$\begin{cases} v \in H^1(\Omega_\epsilon), \quad v = \psi \quad \text{on } \partial\Omega_\epsilon \cap \Omega \quad \text{and} \\ \displaystyle\int_{\Omega_\epsilon} b(X)\nabla v.\nabla\zeta = 0 \qquad \forall\zeta \in H^1(\Omega_\epsilon), \quad \zeta = 0 \quad \text{on } \partial\Omega_\epsilon \cap \Omega. \end{cases} \tag{2.8}$$

Since $(v-Q)^+$ is a test function for (2.8), one easily verifies that $v \leq Q$ in Ω_ϵ. Now by taking $\zeta = (\psi - v)^+$ as a test function in (2.7) and (2.8) and subtracting the two quantities from each other, we obtain

$$\int_{\Omega_\epsilon} b(X)\nabla(\psi - v).\nabla(\psi - v)^+ \leq 0,$$

from which we deduce that $(\psi-v)^+$ is constant in Ω_ϵ. But $(\psi-v)^+ = 0$ on $\partial\Omega_\epsilon \cap \Omega$ leads to $\psi \leq v$ in Ω_ϵ. So $v(E) = Q$ since $Q = \psi(E) \leq v(E) \leq Q$. Hence v takes its maximum at the point E. By the strong maximum principle, we have either $v \equiv Q$ or $v < Q$ in Ω_ϵ. The first assumption is impossible since it leads to $\psi = Q$ on $\partial\Omega_\epsilon \cap \Omega$ which is in contradiction with $\psi < Q$ in $\Omega_{2\epsilon} \supset \partial\Omega_\epsilon \cap \Omega$. Thus $v < Q$ in Ω_ϵ and we get a contradiction with $\psi(E) = Q$ and $b(E)\nabla v.\nu = 0$. □

Proposition 2.9. *Assume that the interior sphere condition is satisfied at each point of* $\overset{\frown}{BB_1} \cup \overset{\frown}{B_1^*A}$. *Then we have*

$$\psi < Q \quad \textit{on} \quad \overset{\frown}{BB_1} \cup \overset{\frown}{B_1^*A} \qquad \textit{and} \qquad \widetilde{\gamma} = 0 \quad \textit{a.e. in} \quad \overset{\frown}{B_1^*A}\,.$$

Proof. First note that by Theorem 2.1 and Proposition 2.8, we have $\psi < Q$ on $\overset{\frown}{B_1^*A}$, which leads to $\widetilde{\gamma} = 0$ a.e. in $\overset{\frown}{B_1^*A}$.
Next let $\zeta \in \mathcal{D}(\Omega_0 \cup \overset{\frown}{BB_1})$. Then ζ is a test function for (P) and we have by taking into account Theorem 2.1 and the fact that $\zeta = 0$ on $\overset{\frown}{B_1T} \cup \overset{\frown}{TA}$

$$\int_{\Omega_0} b(X)\nabla\psi.\nabla\zeta \;=\; 0.$$

It follows that $\psi < Q$, $div(b(X)\nabla\psi) = 0$ in Ω_0, and $b(X)\nabla\psi.\nu = 0$ on $\overset{\frown}{BB_1}$. Hence ψ cannot achieve its maximum value on $\overset{\frown}{BB_1}$ and therefore $\psi < Q$ on $\overset{\frown}{BB_1}$. □

Proposition 2.10. *Assume that* $E \in Int(\overset{\frown}{TB_1^*})$ *and that the interior sphere condition is satisfied at the point* E.

$$\textit{If} \qquad \psi(E) = Q \qquad \textit{then} \qquad \psi \equiv Q \quad \textit{in} \quad \Omega \cap [y > y_E].$$

Proof. By Proposition 2.8, there exists a sequence $E_n \in \Omega$ such that $E_n \longrightarrow E$ and $\psi(E_n) = Q$. By Proposition 2.6, it follows that $\psi(x, y_{E_n}) = Q$ if $x \geq x_{E_n}$. Letting $n \to \infty$, we get $\psi(x, y_E) = Q$ for $x \geq x_E$, which means that $\psi = Q$ on l_E. We deduce then from Proposition 2.7 that $\psi = Q$ in $\Omega \cap [y > y_E]$. □

3 Study of the Free Boundary

This section is devoted to the study of the free boundary which is, by Theorem 2.1, located above the level $y = h$. Throughout the section we denote a solution of (P) by $(\psi, \gamma, \widetilde{\gamma})$.

Proposition 2.6 allows us to define the function $\phi : (h, y_T) \longrightarrow \mathbb{R}$ by

$$\phi(y) = \begin{cases} \sup\{x \in (a(y), b(y)) \ / \psi(x, y) < Q\} & \text{if this set is not empty} \\ a(y) & \text{otherwise.} \end{cases}$$

Then we have

Theorem 3.1. *ϕ is continuous on (h, y_T).*

Proof. Let $\Omega_1 = \Omega \cap [y > h]$, $u = Q - \psi$, $\chi = 1 - \gamma$. Arguing as in the beginning of the proof of Theorem 2.1, one can verify that

$$(\widehat{P}) \begin{cases} \displaystyle\int_{\Omega_1} (b(X)\nabla u + \chi e_x)\nabla\zeta = 0 \qquad \forall \zeta \in H^1(\Omega_1), \quad \zeta = 0 \quad \text{on } \partial\Omega_1 \backslash \, \widehat{B_1 T} \\ \\ u \geq 0 \quad \text{in } \Omega_1, \quad \chi \in H(u) \text{ a.e. in } \Omega_1. \end{cases}$$

Problem $(\widehat{P})$ belongs to a class of problems studied in [11] where the continuity of the free boundary was established in a more general framework. □

As a consequence of Theorem 3.1, we obtain

Proposition 3.1. $\gamma = \chi([\psi = Q])$ *a.e. in* Ω.

Proof. Indeed, we have $\Omega = \Omega_0 \cup \Omega_1 \cup [y = h]$ and we know from Theorem 2.1 and the definition of ϕ, that $\gamma = 0$ a.e. in $\Omega_0 \cup (\Omega_1 \cap [x < \phi(y)])$.
Now let $(x_0, y_0) \in [x > \phi(y)]$. By continuity, there exists $B_r(x_0, y_0) \subset [x > \phi(y)]$ such that $\psi = Q$ on B_r. This leads to $\psi = Q$ in $Z = B_r \cup \big(((x_0, +\infty) \times (y_0 - r, y_0 + r)) \cap \Omega\big)$. From (2.1), we deduce that $\gamma = \gamma(y)$ a.e. in Z. Now if we take $\zeta \in H^1(Z)$ in $(P)i)$ such that $\zeta = 0$ on $\partial Z \cap \Omega$, we obtain

$$\int_Z \gamma(y)\zeta_x = \int_{\widehat{B_1 T}} \gamma(y)\zeta\nu_x = \int_{\widehat{B_1 T}} \zeta\nu_x \qquad \Rightarrow \qquad \gamma = 1 \quad \text{a.e. in } Z.$$

Hence $\gamma = 1$ a.e. in $[x > \phi(y)]$. Now, since the sets $[y = h]$ and $[x = \phi(y)]$ are of Lebesgue measure zero, we get the result. □

Proposition 3.2. *We have* $\psi < Q$ *on* $[y = h] \cap \Omega$.

Proof. Assume that there exists $x_0 \in (x_{B_1^*}, x_{B_1})$ such that $\psi(x_0, h) = Q$. Then $\psi(x, h) = Q$ for all $x \geq x_0$. We claim that there exists a sequence of points $(x_n, y_n)_n$ converging to (x_0, h) and satisfying

$$\forall n \geq 1 \quad x_n \geq x_0, \quad y_n > h \qquad \text{and} \qquad \psi(x_n, y_n) = Q.$$

Indeed, otherwise we would have $\psi < Q$ above the segment $(x_0, x_{B_1}) \times \{h\}$. Then if we consider a small ball $B \subset \Omega$ centred at a point of this segment, we get $\gamma = 0$ a.e. in B since we have also $\psi < Q$ in $[y < h]$. So $div(b(X)\nabla\psi) = 0$ in B and we get a contradiction to the maximum principle.
Using Proposition 2.6, we have $\psi(x, y_n) = Q$ for all $x \geq x_n$.
Arguing as in the proof of the continuity of ϕ (see [11]), one can prove that for some $\eta > 0$ small enough and some n large enough, we have

$$\psi \equiv Q \qquad \text{in} \qquad \big((x_n + \eta, +\infty) \times (h, y_n)\big) \cap \Omega.$$

Consider now a small ball B centred at a point of the segment $(x_n + \eta, x_{B_1}) \times \{h\}$ and such that $B \cap [y > h] \subset \big((x_n + \eta, +\infty) \times (h, y_n)\big) \cap \Omega$. Since $\gamma = 0$ a.e. in $B \cap [y < h]$ and $\gamma = 1$ a.e. in $B \cap [y > h]$, we have $\gamma_x = 0$ in $\mathcal{D}'(B)$. Using (2.1), we obtain $div(b(X)\nabla\psi) = 0$ in $\mathcal{D}'(B)$ and we get a contradiction to the maximum principle. Hence we have $\psi < Q$ on $[y = h] \cap \Omega$. □

Proposition 3.3. *The set* $[\psi < Q]$ *is connected by arcs.*

Proof. Since $\Omega \cap [y \leq h] \subset [\psi < Q]$ is connected, it is enough to show that each point of $[\psi < Q] \cap [y > h]$ can be connected to a point of the line segment $l_{B_1} = [y = h] \cap \Omega$.
Let $(x_0, y_0) \in [\psi < Q] \cap [y > h]$. By Proposition 2.6, we have $\psi(x, y_0) < Q \quad \forall x \leq x_0$. Moreover by Proposition 2.10, one has $\psi(a(y), y) < Q \quad \forall y \in [h, y_0]$. By continuity of ψ, there exists $\epsilon > 0$ small enough such that $\psi(x, y) < Q \quad \forall y \in [h, y_0]$ and $\forall x \in [a(y), a(y) + \epsilon]$. Therefore (x_0, y_0) is connected by the piecewise smooth curve $[a(y_0) + \epsilon, x_0] \times \{y_0\} \cup \{(a(y) + \epsilon, y) : \quad y \in [h, y_0]\}$ to the point $(a(h) + \epsilon, h)$ of l_{B_1}. □

Proposition 3.4. $\displaystyle\lim_{y \to h^+} \phi(y) = x_{B_1} = b(h).$

Proof. Let $\epsilon > 0$. Since $\psi(x_{B_1} - \epsilon, h) < Q$, there exists, by continuity of ψ, $\eta_1 > 0$ such that $\psi(x_{B_1} - \epsilon, y) < Q \quad \forall y \in [h, h + \eta_1]$.
By definition of ϕ, we obtain $\phi(y) > x_{B_1} - \epsilon \quad \forall y \in [h, h + \eta_1]$.
On the other hand, we get from the continuity of $b(y)$ at h, the existence of $\eta_2 > 0$ such that $b(y) < x_{B_1} + \epsilon \quad \forall y \in [h, h + \eta_2]$. Therefore

$$x_{B_1} - \epsilon < \phi(y) \leq b(y) < x_{B_1} + \epsilon \qquad \forall y \in [h, h + \min(\eta_1, \eta_2)].$$

$\square$

4 Uniqueness of the Solution

In this section we establish the uniqueness of the solution by techniques similar to those in [9], [14].

Theorem 4.1. *The solution of the problem (P) is unique.*

To prove Theorem 4.1, we need two lemmas.

Lemma 4.1. *Let $(\psi_1, \gamma_1, \widetilde{\gamma}_1)$, $(\psi_2, \gamma_2, \widetilde{\gamma}_2)$ be two solutions of (P). Then we have*

$$\mathcal{T}(\zeta) = \int_\Omega (b(X)\nabla(\psi_1 - \psi_2)^+ + (\gamma_1 - \gamma_2)^+ e_x).\nabla\zeta \leq \int_J \zeta(\phi_2(y), y)dy$$

$\forall \zeta \in H^1(\Omega) \cap C(\overline{\Omega})$, $\quad \zeta \geq 0 \quad$ *with* $\quad J = \{\, y \in (h, y_T) : \quad \phi_1(y) < \phi_2(y) \,\}$.

Proof. Let $\zeta \in H^1(\Omega) \cap C(\overline{\Omega})$, $\zeta \geq 0$. For $\epsilon > 0$ small enough, take $\xi = \min\Big(\dfrac{(\psi_1 - \psi_2)^+}{\epsilon}, \zeta\Big)$ as a test function in $(P)i)$ for $(\psi_1, \gamma_1, \widetilde{\gamma}_1)$ and for $(\psi_2, \gamma_2, \widetilde{\gamma}_2)$. By subtracting the obtained equations, we get, since $(\psi_1 - \psi_2)(\gamma_1 - \gamma_2) \geq 0$ a.e. in Ω, $(\psi_1 - \psi_2)(\widetilde{\gamma}_1 - \widetilde{\gamma}_2) \geq 0$ and $\nu_x \leq 0$ a.e. on $\widehat{TA}$,

$$\begin{aligned} &\int_{\Omega \cap [(\psi_1 - \psi_2)^+ \geq \epsilon\zeta]} b(X)\nabla(\psi_1 - \psi_2)^+.\nabla\zeta + \int_\Omega (\gamma_1 - \gamma_2)^+ e_x.\nabla\xi \\ &= -\frac{1}{\epsilon}\int_{\Omega \cap [(\psi_1 - \psi_2)^+ < \epsilon\zeta]} b(X)\nabla(\psi_1 - \psi_2)^+.\nabla(\psi_1 - \psi_2)^+ + \int_{\widehat{TA}} (\widetilde{\gamma}_1 - \widetilde{\gamma}_2)^+ \xi\nu_x \leq 0. \end{aligned}$$

Since $\xi = \zeta - \Big(\zeta - \dfrac{(\psi_1 - \psi_2)^+}{\epsilon}\Big)^+$, we obtain

$$\begin{aligned} &\int_{\Omega \cap [(\psi_1 - \psi_2)^+ \geq \epsilon\zeta]} b(X)\nabla(\psi_1 - \psi_2)^+.\nabla\zeta + \int_\Omega (\gamma_1 - \gamma_2)^+ e_x.\nabla\zeta \\ &\qquad \leq \int_\Omega (\gamma_1 - \gamma_2)^+ \Big(\zeta - \frac{(\psi_1 - \psi_2)^+}{\epsilon}\Big)^+_x = I_\epsilon. \end{aligned} \tag{4.1}$$

Using Proposition 3.1, we have

$$I_\epsilon \;=\; \int_{[x < \phi_2(y)] \cap [x \geq \phi_1(y)]} \Big(\zeta - \frac{(Q - \psi_2)^+}{\epsilon}\Big)^+_x$$

$$= \int_J \Big(\int_{\phi_1(y)}^{\phi_2(y)} \Big(\zeta - \frac{(Q-\psi_2)^+}{\epsilon} \Big)_x^+ dx \Big) dy \leq \int_J \zeta(\phi_2(y), y) dy.$$

The lemma follows by letting $\epsilon \to 0$ in (4.1). $\square$

Lemma 4.2. *Let $(\psi_1, \gamma_1, \widetilde{\gamma}_1)$, $(\psi_2, \gamma_2, \widetilde{\gamma}_2)$ be two solutions of (P). Then we have*

$$\mathcal{T}(\zeta) = 0 \qquad \forall \zeta \in H^1(\Omega).$$

Proof. Let $\zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$. For $\delta > 0$ small enough, we consider

$$\alpha_\delta(X) = \Big(1 - \frac{d(X,U)}{\delta}\Big)^+, \qquad \text{with } U = [\psi_1 < Q] = [x < \phi_1(y)].$$

We have $\mathcal{T}(\zeta) = \mathcal{T}(\alpha_\delta \zeta) + \mathcal{T}((1-\alpha_\delta)\zeta)$ and by Lemma 4.1

$$\mathcal{T}(\alpha_\delta \zeta) \leq \int_J \alpha_\delta(\phi_2(y), y) \zeta(\phi_2(y), y) dy.$$

But for $y \in J$, $\phi_2(y) > \phi_1(y)$ and then $(\phi_2(y), y) \notin \overline{U} = \overline{[x < \phi_1(y)]}$ $\forall y \in J$. So $d((\phi_2(y), y), U) > 0$ and $\lim\limits_{\delta \to 0} \alpha_\delta(\phi_2(y), y) = 0$. We deduce that $\limsup\limits_{\delta \to 0} \mathcal{T}(\alpha_\delta \zeta) \leq 0$. Next, we have for $\psi_0 = \min(\psi_1, \psi_2)$, $\gamma_0 = \min(\gamma_1, \gamma_2)$,

$$\begin{aligned}
\mathcal{T}((1-\alpha_\delta)\zeta) &= \int_\Omega (b(X)\nabla(\psi_1 - \psi_0) + (\gamma_1 - \gamma_0)e_x).\nabla((1-\alpha_\delta)\zeta) \\
&= \int_{\Omega \setminus \overline{U}} (b(X)\nabla\psi_1 + \gamma_1 e_x).\nabla((1-\alpha_\delta)\zeta) \\
&\quad - \int_{\Omega \setminus \overline{U}} (b(X)\nabla\psi_0 + \gamma_0 e_x).\nabla((1-\alpha_\delta)\zeta).
\end{aligned}$$

Note that on $[x > \phi_1(y)] = \Omega \setminus \overline{U}$, we have $\psi_1 = Q$, $\gamma_1 = 1$, $\psi_0 = \psi_2$ and $\gamma_0 = \gamma_2$. Moreover $(1-\alpha_\delta)\zeta$ is a test function for (P) since $\overset{\frown}{AB} \subset \overline{U}$. Then we obtain

$$\begin{aligned}
\mathcal{T}((1-\alpha_\delta)\zeta) &= \int_\Omega ((1-\alpha_\delta)\zeta)_x - \int_\Omega (b(X)\nabla\psi_2 + \gamma_2 e_x).\nabla((1-\alpha_\delta)\zeta) \\
&= \int_{\partial\Omega} (1-\alpha_\delta)\zeta\nu_x - \int_{\overset{\frown}{TA}} \widetilde{\gamma}_2 (1-\alpha_\delta)\zeta\nu_x - \int_{\overset{\frown}{B_1 T}} (1-\alpha_\delta)\zeta\nu_x \\
&= \int_{\overset{\frown}{TA}} (1-\widetilde{\gamma}_2)(1-\alpha_\delta)\zeta\nu_x \leq 0
\end{aligned}$$

since $\alpha_\delta = 1$ on $\overset{\frown}{AB} \cup \overset{\frown}{BB_1}$. Hence $\mathcal{T}(\zeta) \leq 0 \quad \forall \zeta \in C^1(\bar{\Omega}), \quad \zeta \geq 0$.

Now let $\zeta \in C^1(\bar{\Omega})$ and $M = \sup_{\bar{\Omega}}|\zeta|$. Note that we have $M \pm \zeta \in C^1(\bar{\Omega})$ and $M \pm \zeta \geq 0$ in Ω. It follows that $\mathcal{T}(M \pm \zeta) \leq 0$ and therefore $\mathcal{T}(\zeta) = 0 \quad \forall \zeta \in C^1(\bar{\Omega})$. Finally, since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$, Lemma 4.2 follows. □

Proof of Theorem 4.1. Starting from Lemma 4.2, we establish as in [14] that

$$\psi_1 = \psi_2 \qquad \text{in} \quad [\psi_1 < Q] \cap [\psi_2 < Q] = [\psi_1 < Q] = [\psi_2 < Q].$$

Therefore $\psi_1 = \psi_2$ in Ω and by Proposition 3.1, we obtain $\gamma_1 = \gamma_2$ a.e. in Ω. Finally, let $\zeta \in H^1(\Omega)$, $\zeta = 0$ on $\partial\Omega \setminus \overset{\frown}{TA}$. Writing the equations satisfied by the two solutions and subtracting them from each other, we obtain $\int_{\overset{\frown}{TA}} (\widetilde{\gamma}_1 - \widetilde{\gamma}_2)\zeta\nu_x = 0$ which leads to $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$ a.e. in $\overset{\frown}{TA}$. □

5 Conclusion

To the best of our knowledge, no regularity result for the free boundary has been obtained from the classical formulation in the absence of monotonicity of the permeability. This is the main motivation for this work in which we established, without any monotonicity of the permeability, that the free boundary is a curve of a continuous function $x = \phi(y)$. Finally it would be quite interesting to study the equivalence of the two formulations.

Acknowledgments

We would like to thank KFUPM for its continuing support.

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