On the Continuity of the Free Boundary in Problems of type $div(a(x)\nabla u) = -(h(x)\chi(u))_{x_1}$

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Abstract

We consider a class of two dimensional free boundary problems including the heterogeneous dam, lubrication and aluminium electrolysis problems. We prove the Lipschitz continuity of the solution and the continuity of the free boundary.

Introduction

Many free boundary problems are described by the following weak formulation

$$(P)\begin{cases} \text{Find } (u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that :} \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \text{ a.e. in } \Omega \\ (ii) \quad u = \varphi \quad \text{on } \Gamma_1 \\ (iii) \quad \displaystyle\int_\Omega \big(a(x)\nabla u + \chi H(x)\big).\nabla\xi dx \leq 0 \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \text{ on } \Gamma_1, \quad \xi \geq 0 \text{ on } \Gamma_2 \end{cases}$$

where Ω is a bounded domain, $a(x) = (a_{ij}(x))$ is a 2-by-2 matrix, $x = (x_1, x_2)$, $H(x)$ is a vector function, Γ_1 and Γ_2 are parts of the boundary $\partial\Omega$ of Ω.

Indeed if $a(x)$ is the permeability of a porous medium Ω and if $H(x) = a(x)e_2$, where $e_2 = (0,1)$, then (P) is the weak formulation of the heterogeneous dam problem with Dirichlet boundary conditions (see [1], [11]).

When $a(x) = h^3(x)I_2$ and $H(x) = h(x)e_2$, where I_2 is the 2-by-2 identity matrix and $h(x)$ a scalar function related to the Reynolds equation, then we have the weak formulation of the lubrication problem (see [4]).

A third model corresponds to $a(x) = k(x)I_2$ and $H(x) = h(x)e_1$, where $e_1 = (1,0)$, $k(x)$ and $h(x)$ are scalar functions. It corresponds to the aluminium electrolysis problem (see

[5]). In this case we obtain, after a suitable change of variables, a similar formulation to (P) (see [7]).

In these problems we are interested to study the free boundary $\Gamma_f = \partial[u > 0] \cap \Omega$ separating two different regions. In the case of the dam and lubrication problems, it separates the region containing the fluid from the rest of the domain. In the case of the aluminium electrolysis problem, the free boundary separates the regions containing liquid and solid aluminium.
The regularity of Γ_f has been studied by many authors in different situations. In [2], H.W. Alt proved that it is an analytic curve $x_2 = \Phi(x_1)$ when $a(x) = I_2$ and $H(x) = e_2$.

In [11], A. Lyaghfouri proved that Γ_f is a continuous curve $x_2 = \Phi(x_1)$ provided that $H(x) = a(x)e_2$, $a_{12}(x) = 0$ and $\dfrac{\partial a_{22}}{\partial x_2} \geq 0$ in $\mathcal{D}'(\Omega)$. Recently, this result was extended in [8] to the case where $div\big(a(x)e_2\big) \geq 0$. Γ_f was shown to be locally represented by continuous curves.

In [6], M. Chipot considered the case where $H(x) = h(x)e_1$, $h(x) \in L^\infty(\Omega)$, and $h_{x_1} \geq 0$ in $\mathcal{D}'(\Omega)$. Then under the following assumptions :

- (A1) $\quad a_{21}\dfrac{h}{a_{11}}$ is Lipschitz continuous, nondecreasing in x_2,

for any $\alpha > x_1$, the function

- (A2) $\quad a_{12}\displaystyle\int_{x_1}^{\alpha} \Big(\frac{h}{a_{11}}\Big)_{x_2}(\xi, x_2)d\xi$ is Lipschitz continuous and non-increasing in x_1,
- (A3) $\quad a_{22}\displaystyle\int_{x_1}^{\alpha} \Big(\frac{h}{a_{11}}\Big)_{x_2}(\xi, x_2)d\xi$ is Lipschitz continuous and non-increasing in x_2,

and for any $\alpha < x_1$, the function

- (A4) $\quad a_{21}\displaystyle\int_{\alpha}^{x_1} \Big(\frac{h}{a_{11}}\Big)_{x_2}(\xi, x_2)d\xi$ is nonnegative, Lipschitz continuous and non-increasing in x_1,
- (A5) $\quad a_{22}\displaystyle\int_{\alpha}^{x_1} \Big(\frac{h}{a_{11}}\Big)_{y}(\xi, x_2)d\xi$ is Lipschitz continuous and non-increasing in x_2,

he proved that Γ_f is a continuous curve $x_1 = \Phi(x_2)$.

In this paper, we would like to consider the problem studied in [6] with the objective of removing the technical assumptions $(A.1)-(A.5)$, that we believe impose unnecessary relationships between h and the matrix a. We shall replace them by the two conditions (2.8)-(2.9).
First we prove that any solution is locally Lipshitz continuous. Then by extending techniques developed in [3], we establish the continuity of the corresponding free boundary.

1 Statement of the problem and reminder of some results

Let Ω be the open bounded domain of $\mathbb{R}^2$ defined by

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ / \ x_2 \in (a_0, b_0), \ \gamma_1(x_2) < x_1 < \gamma_2(x_2)\}$$

where γ_1 and γ_2 are two Lipschitz continuous functions from (a_0, b_0) into $\mathbb{R}$. We set

- $\Gamma_1 = \{(\gamma_1(x_2), x_2) \ / \ x_2 \in (a_0, b_0)\}$
- $\Gamma_2 = \{(\gamma_2(x_2), x_2) \ / \ x_2 \in (a_0, b_0)\}$
- $\Gamma_3 = \partial\Omega \setminus \big(\Gamma_1 \cup \Gamma_2\big)$.

Let $a = (a_{ij})$ be a two-by-two matrix with

$$a_{ij} \in L^\infty(\Omega), \quad |a(x)| \leq M, \quad \text{for a.e. } x \in \Omega, \tag{1.1}$$
$$a(x)\xi.\xi \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad \text{for a.e. } x \in \Omega, \tag{1.2}$$

where λ and M are positive constants.
Let h be a function satisfying for some positive constants $\bar{h} \geq \underline{h}$ and $p > 2$

$$\underline{h} \leq h(x) \leq \bar{h} \quad \text{for a.e. } x \in \Omega \tag{1.3}$$
$$h_{x_1} \in L^p_{loc}(\Omega) \tag{1.4}$$
$$h_{x_1}(x) \geq 0 \quad \text{for a.e. } x \in \Omega. \tag{1.5}$$

We are interested to study the following problem (see [6])

$$(P)\begin{cases} \text{Find } (u, \chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that } : \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(\chi - 1) = 0 \ \text{ a.e. in } \Omega \\ (ii) \quad \displaystyle\int_\Omega \big(a(x)\nabla u + \chi h(x) e_1\big).\nabla\xi dx \ \leq \ 0 \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \text{ on } \Gamma_1 \cup \Gamma_3, \quad \xi \geq 0 \text{ on } \Gamma_2. \end{cases}$$

Remark 1.1. *i) For the existence of a solution of (P), we refer to [1].*

ii) If for $\zeta \in \mathcal{D}(\Omega)$, one takes $\pm\zeta$ as test functions in $(P)ii)$, one gets $div\big(a(x)\nabla u\big) = -(h\chi)_{x_1}$ in $\mathcal{D}'(\Omega)$. Since $h\chi \in L^p_{loc}(\Omega)$ and due to (1.1)-(1.2), it follows (see [9] Theorem 8.24, p. 202) that $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. As a consequence the set $[u > 0]$ is open.

iii) Now we have $div\big(a(x)\nabla u\big) = -h_{x_1}$ in $\mathcal{D}'([u > 0])$. So if $a \in C^{0,\alpha}_{loc}(\Omega)$ $(0 < \alpha < 1)$, we deduce (see [9] Corollary 8.36 and the Remark just after, p. 212) that $u \in C^{1,\alpha}_{loc}([u > 0])$.

In the following we recall some of the properties of the solutions of (P) established in [6]. Actually the Propositions 1.1-1.3 are the equivalent of Propositions 2.1, Corollary 2.4 and Proposition 2.5 of [6] respectively. Proposition 1.4 is the equivalent of Propositions 3.1 and 3.2. Finally Lemma 1.1 is Lemma 3.4 of [6].

Proposition 1.1. *Let (u,χ) be a solution of (P). We have*

$$\chi_{x_1} \leq 0 \qquad in \quad \mathcal{D}'(\Omega). \tag{1.6}$$

Remark 1.2. *In [6], (1.6) is proved assuming that $h \in H^1(\Omega)$ and that h_{x_1} is nonnegative. Actually one can verify easily that (1.6) remains valid if one assumes only that h_{x_1} belongs to $L^1_{loc}(\Omega)$ and is nonnegative, which is ensured by (1.4)-(1.5). Indeed using only the fact that $h_{x_1} \in L^1_{loc}(\Omega)$ and the nonnegativity of h_{x_1}, the author showed in [6] that*

$$\int_\Omega \chi(h\xi)_{x_1} \geq 0 \quad \forall \xi \in \mathcal{D}(\Omega),\ \xi \geq 0.$$

By approximation this inequality remains true for any nonnegative function ξ with compact support in Ω and such that $\xi_{x_1} \in L^2(\Omega)$. Therefore one can take $\xi = \frac{\zeta}{h}$ for any $\zeta \in \mathcal{D}(\Omega)$, $\zeta \geq 0$ and conclude as in [6].

Proposition 1.2. *Let (u,χ) be a solution of (P) and $x_0 = (x_{01}, x_{02}) \in \Omega$.*

i) If $u(x_0) > 0$, then there exists $\epsilon > 0$ such that

$$u(x_1,x_2) > 0 \qquad \forall (x_1,x_2) \in C_\epsilon = B_\epsilon(x_0) \cup \{(x_1,x_2) \in \Omega \,/\, |x_2 - x_{02}| < \epsilon,\ x_1 < x_{01}\}$$

where $B_\epsilon(x_0)$ is the ball centered at x_0 with radius r.

ii) If $u(x_0) = 0$, then $u(x_1, x_{02}) = 0 \quad \forall x_1 \geq x_{01}$.

We then define the function Φ by

$$\Phi(x_2) = \begin{cases} \gamma_1(x_2) & \text{if} \quad \{x_1 \,/\, (x_1,x_2) \in \Omega, \quad u(x_1,x_2) > 0\} = \emptyset \\ \sup\{x_1 \,/\, (x_1,x_2) \in \Omega, \quad u(x_1,x_2) > 0\} & \text{otherwise.} \end{cases} \tag{1.7}$$

Φ is well defined and satisfies

Proposition 1.3. *Φ is lower semi-continuous on (a_0, b_0) and*

$$[u(x_1,x_2) > 0] = [x_1 < \Phi(x_2)].$$

Proposition 1.4. *Let (u,χ) be a solution of (P). Let $x_0=(x_{01},x_{02})\in\Omega$ and $r>0$ such that $B_r(x_0)\subset\Omega$. Then we cannot have the following situations in $B_r(x_0)$*

$$\begin{array}{llllll} (i) & u(x)=0 & \textit{for } x_2=x_{02} & \textit{and} & u(x)>0 & \textit{for } x_2\neq x_{02}. \\ (ii) & u(x)=0 & \textit{for } x_2\geq x_{02} & \textit{and} & u(x)>0 & \textit{for } x_2< x_{02}. \\ (iii) & u(x)>0 & \textit{for } x_2> x_{02} & \textit{and} & u(x)=0 & \textit{for } x_2\leq x_{02}. \end{array}$$

Lemma 1.1. *Let (u,χ) be a solution of (P). Let $(\underline{x}_1,x_{12}),(\underline{x}_1,x_{22})\in\Omega$ with $x_{12}<x_{22}$ and $u(\underline{x}_1,x_{i2})=0$ for $i=1,2$. Let $D=\big((\underline{x}_1,+\infty)\times(x_{12},x_{22})\big)\cap\Omega$. Then we have*

$$\int_D \big(a(x)\nabla u+\chi h(x)e_1\big).\nabla\zeta dx\leq 0$$
$$\forall\zeta\in H^1(D)\cap L^\infty(D),\ \zeta\geq 0,\ \zeta(\underline{x}_1,x_2)=0 \ \textit{a.e. } x_2\in(x_{12},x_{22}).$$

From now on, we assume that

$$a\in C^{0,\alpha}_{loc}(\Omega),\quad \alpha\in(0,1) \tag{1.8}$$
$$\exists c_0\in\mathbb{R}\quad/\quad\forall y\in\Omega\quad:\qquad div(a(x)(x-y))\leq c_0\quad \text{in } \mathcal{D}'(\Omega). \tag{1.9}$$

Note that (1.9) is satisfied in particular if $a\in C^{0,1}$ or simply if $div(a(x)e_1)$, $div(a(x)e_2)\in L^\infty(\Omega)$, where e_1 and e_2 are the vectors defined in the introduction.

2 Lipschitz Continuity of u

Given the jump condition along the free boundary, the optimal regularity we can expect for solutions u of (P) is the local Lipschitz continuity which was proved in [1] assuming a and $a^{-1}(h(x)e_1)$ in $C^{0,\alpha}_{loc}(\Omega)$. Here we propose a different approach that extends the one given in [3] for the homogeneous dam problem.

Theorem 2.1. *Let (u,χ) be a solution of (P). Then*

$$u\in C^{0,1}_{loc}(\Omega).$$

First, we prove two Lemmas

Lemma 2.1. *Let $x_0 = (x_{01}, x_{02})$ and $r, d > 0$ such that $B_r(x_0) \subset [u > 0]$, $\overline{B_r(x_0)} \subset B_d(x_0) \subset \Omega$ and $\partial B_r(x_0) \cap \partial[u > 0] \neq \emptyset$. Then we have for some positive constant C depending only on λ, $\bar{h}$, c_0 and d, but not on r*

$$\min_{\partial B_{r/2}(x_0)} u \;=\; \min_{\overline{B_{r/2}(x_0)}} u \;\leq\; C\, r.$$

Proof. First note that we have $\chi = 1$ a.e. in $B_r(x_0)$ and then $div(a(x)\nabla u) = -(h\chi)_{x_1} = -h_{x_1} \leq 0$ in $H^{-1}(B_r(x_0))$. Let now $m = \min_{\overline{B_{r/2}(x_0)}} u$ and $v = u - m$. Then v satisfies $v \geq 0$ in $B_{r/2}(x_0)$, $v \in H^1(B_{r/2}(x_0))$ and $div(a(x)\nabla v) \leq 0$ in $H^{-1}(B_{r/2}(x_0))$. Using the weak Harnack inequality (see [10] Theorem 4.15 p. 83) for $f \equiv 0$, one can see that either $v \equiv 0$ or $v > 0$ in $B_{r/2}(x_0)$. This means that either $u \equiv m$ or $u > m$ in $B_{r/2}(x_0)$. In both cases we have $\min_{\partial B_{r/2}(x_0)} u = \min_{\overline{B_{r/2}(x_0)}} u$.

Let $\delta > 0$ such that $B_{r+\delta}(x_0) \subset \Omega$, and v defined by

$$v(x) = k\big(e^{-\mu\rho^2} - e^{-\mu(r+\delta)^2}\big)$$

where $\rho^2 = (x_1 - x_{01})^2 + (x_2 - x_{02})^2$, $k = m/\big(e^{-\mu r^2/4} - e^{-\mu(r+\delta)^2}\big)$, $m = \min_{\partial B_{r/2}(x_0)} u$ and $\mu = \dfrac{\kappa}{r^2}$ with $\kappa > \max\left(2, \dfrac{2c_0}{\lambda}\right)$ and c_0 is the constant in (1.9).

Then one can verify that v satisfies

$$\begin{cases} div(a(x)\nabla v) \geq 0 & \text{in} \quad D = B_{r+\delta}(x_0) \setminus \overline{B_{r/2}(x_0)} \\ v = m & \text{on} \quad \partial B_{r/2}(x_0) \\ v = 0 & \text{on} \quad \partial B_{r+\delta}(x_0) \\ |\nabla v| = 2k\mu\rho e^{-\mu\rho^2} & \text{decreases with respect to } \rho. \end{cases}$$

Indeed we have for $\zeta \in \mathcal{D}(D)$, $\zeta \geq 0$

$$\begin{aligned} \int_D a(x)\nabla v.\nabla\zeta &= \int_D -2\mu k e^{-\mu\rho^2} a(x)(x - x_0).\nabla\zeta \\ &= -2\mu k \int_D a(x)(x - x_0).\nabla(e^{-\mu\rho^2}\zeta) + 2\mu k \int_D \zeta[-2\mu e^{-\mu\rho^2}]a(x)(x - x_0).(x - x_0) \\ &\leq 2\mu k c_0 \int_D \zeta e^{-\mu\rho^2} - 4\mu^2 k\lambda\frac{r^2}{4}\int_D \zeta e^{-\mu\rho^2} \quad \text{by (1.2) and since} \quad |x - x_0|^2 \geq r^2/4 \\ &= \mu k[2c_0 - \mu\lambda r^2]\int_D \zeta e^{-\mu\rho^2} \\ &= \mu k[2c_0 - \lambda\kappa]\int_D \zeta e^{-\mu\rho^2} \leq 0 \quad \text{since} \quad \kappa > \frac{2c_0}{\lambda}. \end{aligned} \tag{2.1}$$

Now since $v \leq u$ on ∂D, $\zeta = (v-u)^+\chi(D) \in H^1_0(\Omega)$, where $\chi(E)$ denotes the characteristic function of the set E. So $\pm\zeta$ are test functions for (P) and we have

$$\int_D (a(x)\nabla u + \chi h(x)e_1).\nabla(v-u)^+ dx = 0. \tag{2.2}$$

Moreover clearly (2.1) can be extended by density to non-negative functions of $H^1_0(D)$. Since $(v-u)^+ \in H^1_0(D)$ and is non-negative, we obtain

$$\int_D a(x)\nabla v.\nabla(v-u)^+ dx \leq 0. \tag{2.3}$$

Subtracting (2.2) from (2.3), we get

$$\int_D a(x)\nabla(v-u).\nabla(v-u)^+ dx - \int_D \chi h(x)(v-u)^+_{x_1} dx \leq 0$$

which can be written

$$\int_D a(x)\nabla(v-u).\nabla(v-u)^+ dx - \int_{D\cap[u=0]} (\chi-1)h(x)v_{x_1} dx$$
$$- \int_D h(x)(v-u)^+_{x_1} dx \leq 0. \tag{2.4}$$

Using (1.2), $(P)i)$ and integrating by part the last term in (2.4), we obtain

$$\lambda\int_D |\nabla(v-u)^+|^2 dx - \int_{D\cap[u=0]} (\chi-1)h(x)v_{x_1} dx \ \leq\ -\int_D h_{x_1}(v-u)^+ dx.$$

This leads by (1.3) and (1.5) to

$$\int_{D\cap[u>0]} |\nabla(v-u)^+|^2 dx \leq \int_{D\cap[u=0]} |\nabla v|\big(\frac{\bar{h}}{\lambda} - |\nabla v|\big)dx.$$

We claim that $\displaystyle\int_{D\cap[u>0]} |\nabla(v-u)^+|^2 dx > 0$. Otherwise we will have in particular $\displaystyle\int_{B_r(x_0)\setminus\overline{B_{r/2}(x_0)}} |\nabla(v-u)^+|^2 dx = 0$ which leads to $\nabla(v-u)^+ = 0$ in $B_r(x_0)\setminus\overline{B_{r/2}(x_0)}$. Since $v \leq u$ on $\partial B_{r/2}(x_0)$, we get $v \leq u$ in $B_r(x_0)\setminus\overline{B_{r/2}(x_0)}$. By continuity one has $v \leq u$ on $\partial B_r(x_0)$. Note that $\partial B_r(x_0)\cap\partial[u>0]$ is not empty by assumption . Moreover it is contained in $[u=0]$, since $\partial B_r(x_0)\cap\partial[u>0] \subset \Omega\cap\partial[u>0] = (\overline{[u>0]}\setminus[u>0])\cap\Omega = \overline{[u>0]}\cap[u=0] \subset [u=0]$. It follows that we have $u(z_0)=0$ for some $z_0 \in \partial B_r(x_0)\cap\partial[u>0]$, which leads to $v(z_0)\leq 0$. But this is impossible since $v>0$ in $D = B_{r+\delta}(x_0)\setminus\overline{B_{r/2}(x_0)} \supset \partial B_r(x_0)$. Hence

$$\int_{D\cap[u=0]} |\nabla v|\big(\frac{\bar{h}}{\lambda} - |\nabla v|\big)dx > 0. \tag{2.5}$$

We claim now that $|\nabla v| < \dfrac{\bar{h}}{\lambda}$ on $\partial B_{r+\delta}(x_0)$. Indeed, if not, we will have $|\nabla v| \geq \dfrac{\bar{h}}{\lambda}$ in D since $|\nabla v|$ is non-increasing with respect to ρ and get a contradiction with (2.5). We deduce that $|\nabla v|_{|_{\partial B_{r+\delta}(x_0)}} = 2k\mu(r+\delta)e^{-\mu(r+\delta)^2} < \dfrac{\bar{h}}{\lambda}$.
Letting $\delta \to 0$, we get

$$m \leq \frac{\bar{h}}{2\kappa\lambda}|1 - e^{3\kappa/4}|r = C\,r.$$

$\square$

Lemma 2.2. *Under the assumptions of Lemma 2.1, we have for a constant $C > 0$ depending only on λ, M, $\bar{h}$, c_0, p and d, but not on r*

$$u(x_0) \,\leq\, C\,r.$$

Proof.
We define $w(x) = \dfrac{u(x_0 + rx)}{r}$ for $x \in B_1$, where B_1 is the open unit ball of center $(0,0)$. It is not difficult to check that

$$div\big(\widetilde{a}(x)\nabla\omega\big) = f(x) \qquad \text{in} \quad B_1, \tag{2.6}$$

with

$$\widetilde{a}(x) = \big(\widetilde{a_{ij}}(x)\big), \quad \widetilde{a_{ij}}(x) = a_{ij}(x_0 + rx)$$
$$f(x) = r(-h_{x_1})(x_0 + rx).$$

Since $\widetilde{a} \in L^\infty(B_1)$ is strictly elliptic, and $f \in L^p(B_1)$, we can apply Theorem 4.17 p. 90 [10] (Moser's Harnack inequality) to (2.6). If we denote by $B_{1/2}$ the open ball of center $(0,0)$ and radius $1/2$, we get for a positive constant C_1 depending only on $\lambda, M,$ and p

$$\max_{B_1} w \,\leq\, C_1\Big(\min_{B_{1/2}} w \,+\, |f|_{L^p(B_1)}\Big)$$

Since $p > 2$ and due to (1.4), we have for some constant C_1' depending only on d

$$\begin{aligned} |f|_{L^p(B_1)} &= \Big(\int_{B_1} r^p h_{x_1}^p(x_0 + rx)dx\Big)^{1/p} = \Big(\int_{B_r} \frac{r^p}{r^2} h_{x_1}^p(y)dy\Big)^{1/p} \\ &= r^{(1-\frac{2}{p})}|h_{x_1}|_{L^p(B_r)} \leq d^{(1-\frac{2}{p})}|h_{x_1}|_{L^p(B_d)} = C_1'(d). \end{aligned}$$

We deduce by using Lemma 2.1

$$\begin{aligned}\frac{1}{r}u(x_0) \quad &\leq \quad \frac{1}{r}\max_{B_1} u(x_0+rx) \;\leq\; C_1\Big(\frac{1}{r}\min_{B_{1/2}} u(x_0+rx) \;+\; C_1'\Big)\\ &= C_1\Big(\frac{1}{r}\min_{B_{r/2}(x_0)} u \;+\; C_1'\Big) = C_1\Big(\frac{m}{r} \;+\; C_1'\Big) \;\leq\; C.\end{aligned}$$

□

Remark 2.1. *If (1.4) is replaced by $h_{x_1} \in L^p(\Omega)$, the constants in Lemmas 2.1 and 2.2 clearly will not depend on d.*

Proof of Theorem 2.1. Let x, $y \in \Omega$. Without loss of generality, one can choose x, y such that

$$|x-y| < d/2 \quad \text{ and } \quad B_{2d}(x),\; B_{2d}(y) \subset \Omega \quad \text{ for some } d > 0.$$

Set $r(z) = \min(d, dist(z,[u=0]))$, where $dist(z,[u=0])$ denotes the distance between z and the set $[u=0]$. Remark that we have $B_{r(z)}(z) \subset [u>0]$ whenever $r(z) > 0$.
Indeed, if $z' \in B_{r(z)}(z)$, we have

$$dist(z',[u=0]) \geq dist(z,[u=0]) - |z-z'| \geq r(z) - |z-z'| > 0.$$

Now if $u(x) = u(y) = 0$, we have $|u(x)-u(y)| = 0 \leq |x-y|$.

If $u(x) > 0$ and $u(y) = 0$, then $y \notin B_{r(x)}(x) \subset [u>0]$.
So $r(x) \leq |x-y| < d/2 < d$ and then $\partial B_{r(x)}(x) \cap \partial[u>0] \neq \emptyset$.
By Lemma 2.2, we obtain $u(x) \leq Cr(x)$. Therefore

$$|u(x)-u(y)| = u(x) \leq Cr(x) \leq C|x-y|.$$

If $u(x) = 0$ and $u(y) > 0$, we conclude as before.

Let us assume $u(x) > 0$ and $u(y) > 0$. We distinguish two cases :

i) $\underline{\frac{1}{2}\max(r(x),r(y)) < |x-y|}$:

Then we have $r(x)$, $r(y) < d$ and $\partial B_{r(x)}(x) \cap \partial[u>0] \neq \emptyset$, $\partial B_{r(y)}(y) \cap \partial[u>0] \neq \emptyset$. Applying Lemma 2.2, we obtain

$$|u(x)-u(y)| \leq u(x)+u(y) \leq C(r(x)+r(y)) \leq 2C\max(r(x),r(y)) \leq 4C|x-y|.$$

ii) $\underline{\frac{1}{2}\max(r(x),r(y)) \geq |x-y| > 0}$:

Assume that $r(x) \geq r(y)$. Then $\dfrac{1}{2}\max(r(x), r(y)) = \dfrac{r(x)}{2} \geq |x - y|$. We distinguish two cases :

$* \; r(x) < \dfrac{d}{2}$.

Let $z \in B_1$. We have for $z' \in [u = 0]$

$$\begin{array}{rcl} d(x + r(x)z, [u = 0]) & \leq & d(x + r(x)z, z') \leq d(x + r(x)z, x) + d(x, z') \\ & = & r(x) \parallel z \parallel + \parallel x - z' \parallel \leq r(x) + \parallel x - z' \parallel \qquad \text{because } z \in B_1. \end{array}$$

Since this holds for arbitrary $z' \in [u = 0]$, we obtain

$$d(x + r(x)z, [u = 0]) \;\leq\; r(x) \;+\; d(x, [u = 0]) \;=\; 2r(x) \;<\; d.$$

So $r(x + r(x)z) < d$ and then $\partial B_{r(x+r(x)z)}(x + r(x)z) \cap \partial[u > 0] \neq \emptyset$. Applying Lemma 2.2, we get

$$u(x + r(x)z) \leq Cr(x + r(x)z) \leq 2Cr(x).$$

We deduce that the function v defined by

$$v(z) = \frac{u(x + r(x)z)}{r(x)}, \qquad z \in B_1$$

is uniformly bounded in B_1 i.e. $v(z) \leq 2C \quad \forall z \in B_1$. Moreover, it satisfies

$$\begin{cases} div(\widetilde{a}(z)\nabla v) = f(z) & \text{in} \quad B_1 \\ v \in C^{1,\alpha}(B_1) & \text{(see [9], Corollary 8.36 p. 212 and the Remark after it)} \end{cases}$$

where

$$\widetilde{a}(z) = \widetilde{a}(x + r(x)z) \qquad \text{and} \qquad f(z) = -r(x)(h_{x_1})(x + r(x)z).$$

Applying Theorem 8.32 p 210 of [9] and taking into account the Remark after Corollary 8.36, we get

$$|v|_{1,\alpha,\overline{B}_{1/2}} \leq C\Big(|v|_{0,B_1} \;+\; |f|_{p,B_1}\Big),$$

where C depends only on $dist(\overline{B}_{1/2}, \partial B_1)$. In particular, $|\nabla v|_{0,\overline{B}_{1/2}}$ is uniformly bounded. Now, since $(y - x)/r(x) \in \overline{B}_{1/2}$, we have

$$\Big|v\Big(\frac{y - x}{r(x)}\Big) - v(0)\Big| \leq C\Big|\frac{y - x}{r(x)}\Big|$$

from which we deduce that

$$|u(y) - u(x)| \leq C|y - x|.$$

$* \ r(x) \geq \dfrac{d}{2}$.

We consider, as above, the function v defined on B_1. Remark that we have

$$|v|_{0,B_1} \leq \frac{|u|_{0,B_d(x)}}{r(x)} \leq \frac{2}{d}|u|_{0,B_d(x)}.$$

Then $|\nabla v|_{0,\overline{B}_{1/2}} \leq C(d)$ and we get by arguing as before

$$|u(y) - u(x)| \leq C(d)|y - x|.$$

$\square$

Remark 2.2. *If in (P)(ii), $h(x)e_1$ is replaced by a vector function $H(x)$ that satisfies*

$$|H(x)| \leq \bar{h} \quad \text{for a.e. } x \in \Omega$$
$$div(H) \in L^p_{loc}(\Omega), \ p > 2$$
$$div(H)(x) \geq 0 \quad \text{for a.e. } x \in \Omega,$$

then one can verify that the above proof can be easily extended to show that the solution u is also locally Lipschitz continuous.

3 A Barrier Function

In this section, we construct a function that will be used to prove the continuity of ϕ by comparing it to u near a free boundary point.

Let $(\underline{x}_1, x_{12})$, $(\underline{x}_1, x_{22}) \in \Omega$ such that $x_{12} < x_{22}$. We assume that $\epsilon = x_{22} - x_{12}$ is small enough to guarantee that

$$(\underline{x}_1, \underline{x}_1 + 2\epsilon) \times (x_{12} - \epsilon, x_{22} + \epsilon) \subset\subset \Omega.$$

Let $Z = (\underline{x}_1, \underline{x}_1 + \epsilon) \times (x_{12} - \epsilon, x_{22} + \epsilon)$. We denote by v the unique solution in $H^1(Z)$ of

$$\begin{cases} div(a(x)\nabla v) = -h_{x_1} & \text{in } Z \\ v = \varphi(x) = \epsilon(\underline{x}_1 + \epsilon - x_1)^+ & \text{on } \partial Z. \end{cases} \tag{3.1}$$

Remark 3.1. *We deduce from (3.1) (see [9], Corollary 8.36 p. 212 and the Remark after it) that $v \in C^{1,\alpha}_{loc}\big(Z \cup \{\underline{x}_1 + \epsilon\} \times (x_{12} - \epsilon, x_{22} + \epsilon)\big)$.*

First we have the following estimate

Proposition 3.1. *There exists a positive constant C independent of ϵ such that*

$$0 < v \leq C\epsilon^{2(1-\frac{1}{p})} \qquad in \quad Z.$$

Proof. Since $div(a(x)\nabla v) = -h_{x_1} \leq 0$ in Z and $v \geq 0$ on ∂Z, we obtain by the weak maximum principle (see [9], Theorem 8.1 p. 179) that $v \geq 0$ in Z.
Now because of the boundary condition, the strong maximum principle (see [9], Theorem 8.19 p. 198) leads to $v > 0$ in Z.

To prove the second inequality, we introduce the function

$$\begin{aligned} \omega \quad : \quad & Y = (0,1) \times (0,3) \longrightarrow \mathbb{R}^+ \\ & x' = (x'_1, x'_2) \longmapsto \omega(x') = v(\underline{x}_1 + \epsilon x'_1, x_{12} - \epsilon + \epsilon x'_2). \end{aligned}$$

Then it is not difficult to check that

$$\begin{cases} div(\widehat{a}(x')\nabla\omega) = -\epsilon^2 \widehat{h_{x_1}} & \text{in } Y \\ \omega = \epsilon^2(1 - x'_1) & \text{on } \partial Y \end{cases} \tag{3.2}$$

where

$$\widehat{a}(x') = a(\underline{x}_1 + \epsilon x'_1, x_{12} - \epsilon + \epsilon x'_2), \quad \widehat{h_{x_1}}(x') = h_{x_1}(\underline{x}_1 + \epsilon x'_1, x_{12} - \epsilon + \epsilon x'_2).$$

Note that we have by (1.1), (1.2) and (1.5)

$$\begin{gathered} \widehat{a}(x')\xi.\xi \geq \lambda|\xi|^2, \qquad \forall \xi \in \mathbb{R}^2, \quad \forall x' \in Y \\ |\widehat{a}(x')| \leq M, \qquad 0 \leq \widehat{h_{x_1}}(x'), \qquad \text{a.e.} \quad x' \in Y. \end{gathered}$$

Applying Theorem 8.16 p. 191 of [9], we get

$$\sup_Y \omega \leq \sup_{\partial Y} \omega + \frac{C_1}{\lambda} |\epsilon^2 \widehat{h_{x_1}}|_{L^{q/2}(Y)}$$

where $q = 2p > 2$ and C_1 is a positive constant depending only on Y. Moreover

$$|\epsilon^2 \widehat{h_{x_1}}|_{L^{q/2}(Y)} = \Big(\int_Y \epsilon^{2p} \widehat{h_{x_1}}^p(x') dx' \Big)^{1/p} = \Big(\int_Z \frac{\epsilon^{2p}}{\epsilon^2} h^p_{x_1}(x) dx \Big)^{1/p} = \epsilon^{2(1-\frac{1}{p})} |h_{x_1}|_{L^p(Z)}.$$

Hence for ϵ small enough

$$\sup_Z v = \sup_Y \omega \leq \epsilon^2 + \frac{C_1}{\lambda} \epsilon^{2(1-\frac{1}{p})} |h_{x_1}|_{L^p(Z)} \leq \epsilon^2 + \frac{C_1}{\lambda} \epsilon^{2(1-\frac{1}{p})} |h_{x_1}|_{L^p(Z')} \leq C\epsilon^{2(1-\frac{1}{p})}$$

where Z' is a subset of Ω that contains $\bar{Z}$ and which does not depend on ϵ. □

Now we have the following gradient estimate

Proposition 3.2. *There exists a positive constant C independent of ϵ such that*

$$|\nabla v(x)| \le C\epsilon^{(1-\frac{2}{p})} \qquad \forall x \in T = \{\underline{x}_1 + \epsilon\} \times [x_{12}, x_{22}].$$

Proof. Let $S = \{1\} \times (\frac{1}{4}, \frac{11}{4})$ and $Y' = (\frac{1}{2}, 1) \times (\frac{1}{2}, \frac{5}{2})$. Since S is a $C^{1,\alpha}$ boundary portion of ∂Y, $\omega = 0$ on S, we deduce from (3.2) by applying Corollary 8.36 p. 212 of [9] that $\omega \in C^{1,\alpha}(Y \cup S)$ with the following estimate

$$|\omega|_{1,\alpha,Y'} \le C\Big(|\omega|_{0,Y} + |\epsilon^2 \widehat{h_{x_1}}|_{p,Y}\Big)$$

where $C = C(\lambda, M, K, d', S)$, $\quad d' = dist(Y', \partial Y \setminus S)$, $\quad K = \max_{i,j}(|a_{ij}|_{0,\alpha,Z'})$. Clearly C is a constant independent of ϵ.
Taking into account the estimate in Proposition 3.1 and the fact that $|\epsilon^2 \widehat{h_{x_1}}|_{p,Y} = \epsilon^{2(1-\frac{1}{p})}|h_{x_1}|_{p,Z} \le \epsilon^{2(1-\frac{1}{p})}|h_{x_1}|_{p,Z'}$, we obtain for another constant independent of ϵ still denoted by C

$$|\nabla\omega|_{0,Y'} \le |\omega|_{1,\alpha,Y'} \le C\epsilon^{2(1-\frac{1}{p})}$$

which leads, in particular, to

$$|\nabla\omega(1, x_2')| \le C\epsilon^{2(1-\frac{1}{p})} \qquad \forall x_2' \in [1, 2].$$

Therefore

$$|\nabla v(\underline{x}_1 + \epsilon, x_2)| = \frac{1}{\epsilon}\Big|\nabla\omega\Big(1, \frac{x_2 - x_{12} + \epsilon}{\epsilon}\Big)\Big| \le C\,\epsilon^{(1-\frac{2}{p})} \qquad \forall x_2 \in [x_{12}, x_{22}].$$

□

The main result of this section is the following Lemma

Lemma 3.1. *For ϵ small enough, we have*

$$\int_D \Big(a(x)\nabla v + \chi([v>0])h(x)e_1\Big).\nabla\zeta \ge 0$$
$$\forall \zeta \in H^1(D), \quad \zeta \ge 0, \quad \zeta = 0 \text{ on } \partial D \setminus \Gamma_2 \tag{3.3}$$

where v is extended by 0 to $D = \Big((\underline{x}_1, +\infty) \times (x_{12}, x_{22})\Big) \cap \Omega$

Proof. Let ν be the outward unit normal vector to D. First we have by Proposition 3.2 and (1.3) $a(x)\nabla v.\nu + h(x)\nu_{x_1} \ge -MC.\epsilon^{(1-\frac{2}{p})} + \underline{h} \ge 0$ on T for ϵ small enough. Next, for $\zeta \in H^1(D)$, $\quad \zeta \ge 0$, $\quad \zeta = 0$ on $\partial D \setminus \Gamma_2$, we have

$$\int_D \Big(a(x)\nabla v + \chi([v>0])h(x)e_1\Big).\nabla\zeta dx$$
$$= \int_{D\cap[v>0]} \Big(a(x)\nabla v + \chi([v>0])h(x)e_1\Big).\nabla\zeta dx$$
$$= -\int_{D\cap[v>0]} \Big(div(a(x)\nabla v) + h_{x_1}(x)\Big)\zeta dx + \int_T \Big(a(x)\nabla v.\nu + h(x)\nu_{x_1}\Big)\zeta d\sigma \geq 0.$$

□

4 Continuity of The Free Boundary

This last section is devoted to the upper semi-continuity of ϕ. The proof is based on comparing u with respect to barrier functions introduced in the previous section.
Using the notations of Section 3, we first prove the following lemma

Lemma 4.1. *Let v be the barrier function defined by* (3.1)*. Let (u,χ) be a solution of (P). Assume that*

$$u(\underline{x}_1, x_2) \leq v(\underline{x}_1, x_2) \quad \forall x_2 \in (x_{12}, x_{22}) \quad \text{and} \quad u(\underline{x}_1, x_{i2}) = 0 \quad i = 1, 2.$$

Then we have

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{D_\delta} a(x)\nabla(u-v)^+.\nabla(u-v)^+ dx \;=\; 0$$

where $D_\delta = D\cap[v>0]\cap[0<u-v<\delta]$.

Proof. For $\delta, \eta > 0$, we consider

$$H_\delta(s) = \min\Big(\frac{s^+}{\delta}, 1\Big), \qquad d_\eta(x_1) = H_\eta(x_1 - \bar{x}_1), \quad \bar{x}_1 = \underline{x}_1 + \epsilon.$$

Then $\zeta = H_\delta(u-v) + d_\eta(1 - H_\delta(u)) \in H^1(D)\cap L^\infty(D)$ is a nonnegative function vanishing on $[x = \underline{x}_1]$. So by Lemma 1.1, we have

$$\int_D \Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla(H_\delta(u-v))dx$$
$$\leq -\int_D \Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla(d_\eta(1 - H_\delta(u)))dx. \tag{4.1}$$

Given that $u(\underline{x}_1, x_{i2}) = 0$ for $i = 1, 2$, we deduce from Proposition 1.2 *ii*) that $u(x_1, x_{i2}) = 0 \quad \forall x_1 \geq \underline{x}_1$, $i = 1, 2$. This leads to $u(x_1, x_{i2}) = 0 \leq v(x_1, x_{i2}) \quad \forall x_1 \geq \underline{x}_1$, $i = 1, 2$. Moreover we have $u(\underline{x}_1, x_2) \leq v(\underline{x}_1, x_2) \quad \forall x_2 \in (x_{12}, x_{22})$. It follows that $u \leq v$ on $\partial D \cap \Omega$, and therefore since $H_\delta(s) = 0$ for $s \leq 0$, we obtain $H_\delta(u - v) = 0$ on $\partial D \cap \Omega = \partial D \setminus \Gamma_2$. Hence we have by (3.3)

$$-\int_D \Big(a(x)\nabla v + \chi([v > 0])h(x)e_1\Big).\nabla(H_\delta(u - v))dx \leq 0. \tag{4.2}$$

Adding (4.1) and (4.2), we get

$$\begin{aligned}\int_D a(x)\nabla(u - v).\nabla(H_\delta(u - v))dx \leq &\int_D h(x)(\chi([v > 0]) - \chi)e_1.\nabla(H_\delta(u - v))dx \\ &- \int_D \Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla(d_\eta(1 - H_\delta(u)))dx\end{aligned}$$

which can be written since $d_\eta = 0$ on $[v > 0]$

$$\begin{aligned}&\int_{D\cap[v>0]} H'_\delta(u - v)a(x)\nabla(u - v).\nabla(u - v)dx \\ &\leq -\int_{D\cap[v=0]} H'_\delta(u)a(x)\nabla u.\nabla u - \int_{D\cap[v=0]} \chi h(x)e_1.\nabla(H_\delta(u))dx \\ &+ \int_{D\cap[v=0]} \Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla((1 - d_\eta)(1 - H_\delta(u)))dx \\ &+ \int_{D\cap[v=0]} \Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla(H_\delta(u))dx \\ &\qquad = I_1^\delta + I_2^\delta + I_3^\delta + I_4^\delta.\end{aligned}$$

Note that $I_1^\delta + I_2^\delta + I_4^\delta = 0$. Moreover

$$\begin{aligned}I_3^\delta = &-\int_{D\cap[v=0]} (1 - d_\eta)\Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla(H_\delta(u))dx \\ &- \int_{D\cap[v=0]} (1 - H_\delta(u))\Big(a(x)\nabla u + \chi h(x)e_1\Big).\nabla d_\eta dx = I_5^\delta + I_6^\delta.\end{aligned}$$

Since $d_\eta \to 1$ a.e. in $D \cap [v = 0]$ when $\eta \to 0$, we obtain by the Lebesgue theorem in $L^1(D \cap [v = 0])$ that $\lim\limits_{\eta\to 0} I_5^\delta = 0$.

Now we have

$$\begin{aligned} I_6^\delta &= -\int_{D\cap[u=v=0]} \chi h(x) e_1.\nabla d_\eta dx \\ &\quad -\int_{D\cap[u>v=0]} (1-H_\delta(u))\Big(a(x)\nabla u + h(x)e_1\Big).\nabla d_\eta dx \\ &= I_7^\delta + I_8^\delta. \end{aligned}$$

Note that

$$I_7^\delta = -\int_{D\cap[u=v=0]} \chi h(x).\partial_{x_1} d_\eta dx = \frac{-1}{\eta}\int_{D\cap[u=v=0]\cap[\bar{x}_1<x_1<\bar{x}_1+\eta]} \chi h(x) dx \leq 0.$$

Since $u \in C^{0,1}_{loc}(\Omega)$, one has for some constant C

$$\begin{aligned} |I_8^\delta| &\leq \frac{C}{\eta}\int_{D\cap[u>v=0]\cap[\bar{x}_1<x_1<\bar{x}_1+\eta]} (1-H_\delta(u))dx \\ &= \frac{C}{\eta}\int_J\int_{\bar{x}_1}^{\min(\phi(x_2),\bar{x}_1+\eta)} (1-H_\delta(u))dx \\ &\leq C\int_J \Big(\frac{1}{\eta}\int_{\bar{x}_1}^{\bar{x}_1+\eta}(1-H_\delta(u))dx_1\Big)dx_2, \end{aligned}$$

where $J = \{x_2 \in (x_{12}, x_{22}) \ / \ \phi(x_2) > \bar{x}_1\}$.
Since the function $x_1 \mapsto 1 - H_\delta(u(x_1, x_2))$ is continuous, we have

$$\lim_{\eta\to 0} f_\eta(x_2) \ = \ 1 - H_\delta(u(\bar{x}_1, x_2)) \qquad \forall x_2 \in (x_{12}, x_{22})$$

where $f_\eta(x_2) \ = \ \dfrac{1}{\eta}\displaystyle\int_{\bar{x}_1}^{\bar{x}_1+\eta}(1-H_\delta(u(x_1,x_2)))dx_1$. Moreover $|f_\eta(x_2)| \leq 1$ for all $x_2 \in (x_{12}, x_{22})$. Then we obtain by using the Lebesgue theorem in $L^1(J)$

$$\lim_{\eta\to 0}\int_J f_\eta(x_2)dx_2 = \int_J (1-H_\delta(u(\bar{x}_1,x_2)))dx_2.$$

So

$$\limsup_{\eta\to 0}|I_8^\delta| \leq C\int_J (1-H_\delta(u))(\bar{x}_1, x_2)dx_2.$$

Hence

$$\int_{D\cap[v>0]\cap[0<u-v<\delta]} \frac{1}{\delta} a(x)\nabla(u-v)^+.\nabla(u-v)^+ dx \leq C\int_J (1-H_\delta(u(\bar{x}_1,x_2)))dx_2.$$

But for $x_2 \in J$, we have $u(\bar{x}_1, x_2) > 0$ and then $\lim_{\delta \to 0} 1 - H_\delta(u(\bar{x}_1, x_2)) = 0$. Letting $\delta \to 0$ in the above inequality, we get the result by the Lebesgue theorem in $L^1(J)$. □

Theorem 4.1.

ϕ is continuous at each $x_2 \in (a_0, b_0)$ such that $(\phi(x_2), x_2) \in \Omega$.

Proof. Let $\epsilon > 0$ small enough. Let $x_{02} \in (a_0, b_0)$. Set $x_0 = (\phi(x_{02}), x_{02}) = (x_{01}, x_{02})$ and assume that $x_0 \in \Omega$. Since $u(x_0) = 0$ and u is continuous, there exists $\eta_1 \in (0, \epsilon)$ such that

$$u(x_1, x_2) \leq \epsilon^2 \qquad \forall (x_1, x_2) \in B_{\eta_1}(x_0). \tag{4.3}$$

By Proposition 1.4, one of the following situations is true

$$\begin{array}{llllll} i) & \exists (x_{11}, x_{12}) \in B_{\eta_1}(x_0) & \text{such that} & x_{12} < x_{02} & \text{and} & u(x_{11}, x_{12}) = 0 \\ ii) & \exists (x_{21}, x_{22}) \in B_{\eta_1}(x_0) & \text{such that} & x_{22} > x_{02} & \text{and} & u(x_{21}, x_{22}) = 0. \end{array}$$

Let us assume that $i)$ holds.
Set $\underline{x}_1 = \max(\phi(x_{02}), x_{11})$ and assume that ϵ is small enough so that

$$(\underline{x}_1 - \epsilon, \underline{x}_1 + 2\epsilon) \times (x_{12} - \epsilon, x_{12} + 2\epsilon) \subset\subset \Omega.$$

Let v_1 be the barrier function defined by (3.1) in the set $Z_1 = (\underline{x}_1, \underline{x}_1 + \epsilon) \times (x_{12} - \epsilon, x_{12} + 2\epsilon)$. We consider the extension by 0 of v_1 to $D_1 = \Big((\underline{x}_1, +\infty) \times (x_{12}, x_{02})\Big) \cap \Omega$ which clearly satisfies (3.3).
Now since $\{\underline{x}_1\} \times (x_{12}, x_{02}) \subset B_{\eta_1}(x_0)$, we have

$$u(\underline{x}_1, x_2) \leq \epsilon^2 = v_1(\underline{x}_1, x_2) \qquad \forall x_2 \in (x_{12}, x_{02}). \tag{4.4}$$

Moreover since $u(\underline{x}_1, x_{12}) = u(\underline{x}_1, x_{02}) = 0$, we get by Proposition 1.2 $ii)$

$$u(x_1, x_{12}) = u(x_1, x_{02}) = 0 \qquad \forall x_1 \geq \underline{x}_1. \tag{4.5}$$

Let $D_1^+ = (\underline{x}_1, \underline{x}_1 + \epsilon) \times (x_{12}, x_{02})$ and $\Delta_1 = (\underline{x}_1 - \epsilon, \underline{x}_1 + \epsilon) \times (x_{12}, x_{02})$. Due to (4.4) one can extend $(u - v_1)^+$ by 0 to $\Delta_1 \setminus D_1^+$ so that $(u - v_1)^+ \in H^1(\Delta_1)$. Then we have for $\zeta \in \mathcal{D}(\Delta_1)$ by the Lebesgue theorem in $L^1(D_1^+)$

$$\begin{aligned} \int_{\Delta_1} a(x) \nabla (u - v_1)^+ . \nabla \zeta dx &= \int_{D_1^+} a(x) \nabla (u - v_1)^+ . \nabla \zeta dx \\ &= \lim_{\delta \to 0} \int_{D_1^+} H_\delta(u - v_1) a(x) \nabla (u - v_1)^+ . \nabla \zeta dx = \lim_{\delta \to 0} I_\delta. \end{aligned}$$

Note that

$$\begin{aligned} I_\delta &= \int_{D_1^+} a(x)\nabla(u-v_1)^+.\nabla(H_\delta(u-v_1)\zeta)dx \\ &\quad -\frac{1}{\delta}\int_{D_1^+\cap[0<u-v_1<\delta]} \zeta a(x)\nabla(u-v_1).\nabla(u-v_1)dx \\ &= I_\delta^1 - I_\delta^2. \end{aligned}$$

By Lemma 4.1, $\lim\limits_{\delta\to 0} I_\delta^2 = 0$, since

$$|I_\delta^2| \le \sup_{D_1^+}|\zeta| \, . \, \frac{1}{\delta}\int_{D_1^+\cap[0<u-v_1<\delta]} a(x)\nabla(u-v_1).\nabla(u-v_1)dx.$$

We claim that $I_\delta^1 = 0$. Indeed, first because $H_\delta(u-v_1) = 0$ whenever $u \le v_1$, we have $\nabla(u-v_1)^+.\nabla(H_\delta(u-v_1)\zeta) = \nabla(u-v_1).\nabla(H_\delta(u-v_1)\zeta)$ a.e. in D_1^+. Therefore

$$I_\delta^1 = \int_{D_1^+} a(x)\nabla u.\nabla(H_\delta(u-v_1)\zeta)dx - \int_{D_1^+} a(x)\nabla v_1.\nabla(H_\delta(u-v_1)\zeta)dx.$$

Since $u \le v_1$ on $\partial D_1^+ \setminus [x_1 = \underline{x}_1 + \epsilon]$, we have $H_\delta(u-v_1) = 0$ on $\partial D_1^+ \setminus [x_1 = \underline{x}_1 + \epsilon]$. Moreover $\zeta = 0$ on $[x_1 = \underline{x}_1 + \epsilon]$. So $H_\delta(u-v_1)\zeta \in H_0^1(D_1^+)$ and therefore from the definition of v_1, we obtain

$$\int_{D_1^+} a(x)\nabla v_1.\nabla(H_\delta(u-v_1)\zeta)dx = -\int_{D_1^+} h(x).(H_\delta(u-v_1)\zeta)_{x_1}dx.$$

Now $\pm H_\delta(u-v_1)\zeta\chi(D_1^+)$ are test functions for (P), $\chi = 1$ a.e. in $D_1^+ \cap [u>0]$ and $H_\delta(u-v_1) = 0$ whenever $u = 0$. So we obtain

$$\begin{aligned} &\int_{D_1^+} a(x)\nabla u.\nabla(H_\delta(u-v_1)\zeta)dx = -\int_{D_1^+} \chi h(x).(H_\delta(u-v_1)\zeta)_{x_1}dx \\ &= -\int_{D_1^+\cap[u>0]} h(x).(H_\delta(u-v_1)\zeta)_{x_1}dx = -\int_{D_1^+} h(x).(H_\delta(u-v_1)\zeta)_{x_1}dx. \end{aligned}$$

Hence $I_\delta^1 = 0$. Consequently

$$\int_{\Delta_1} a(x)\nabla(u-v_1)^+.\nabla\zeta dx \, = \, 0 \qquad \forall\zeta\in\mathcal{D}(\Delta_1)$$

which leads by (4.4) and the strong maximum principle to $(u-v_1)^+ \equiv 0$ in Δ_1. Consequently $u \le v_1$ in D_1^+ and in particular $u(\underline{x}_1+\epsilon, x_2) = 0 \quad \forall x_2 \in (x_{12}, x_{02})$. Therefore

$$u(x_1,x_2) = 0 \qquad \forall x_1 \ge \underline{x}_1 + \epsilon = \bar{x}_1, \qquad \forall x_2 \in [x_{12}, x_{02}].$$

Now, by continuity of u there exists $\eta_2 \in (0, x_{02} - x_{12})$ such that

$$u(x_1, x_2) \leq \epsilon^2 \qquad \forall (x_1, x_2) \in B_{\eta_2}(\bar{x}_1, x_{02}).$$

By Proposition 1.4, there exists $(x_{21}, x_{22}) \in B_{\eta_2}(\bar{x}_1, x_{02})$ such that

$$x_{21} > \bar{x}_1, \qquad x_{22} > x_{02} \qquad \text{and} \qquad u(x_{21}, x_{22}) = 0.$$

Set $\underline{x}'_1 = x_{21}$ and assume that ϵ is small enough so that

$$(\underline{x}'_1, \underline{x}'_1 + 2\epsilon) \times (x_{22} - 2\epsilon, x_{22} + \epsilon) \subset\subset \Omega.$$

Let v_2 be the barrier function defined by (3.1) in the set $Z_2 = (\underline{x}'_1, \underline{x}'_1 + \epsilon) \times (x_{22} - 2\epsilon, x_{22} + \epsilon)$. Clearly the extension by 0 of v_2 to $D_2 = \Big((\underline{x}'_1, +\infty) \times (x_{02}, x_{22})\Big) \cap \Omega$ satisfies (3.3).
Then, since $\{\underline{x}'_1\} \times (x_{02}, x_{22}) \subset B_{\eta_2}(\bar{x}_1, x_{02})$, we have

$$u(\underline{x}'_1, x_2) \leq \epsilon^2 = v_2(\underline{x}'_1, x_2) \qquad \forall x_2 \in (x_{02}, x_{22}).$$

Arguing as above, we deduce that $(u - v_2)^+ \equiv 0$ in $D_2 \cap [v_2 > 0]$. Then

$$u(x_1, x_2) \equiv 0 \qquad \forall x_1 \geq \underline{x}'_1 + \epsilon, \qquad \forall x_2 \in [x_{02}, x_{22}].$$

Hence

$$u(x_1, x_2) \equiv 0 \qquad \forall x_1 \geq \underline{x}'_1 + \epsilon, \qquad \forall x_2 \in [x_{12}, x_{22}].$$

Note that if $ii)$ holds, we argue similarly to obtain the same conclusion. Finally we have proved for all $x_2 \in (x_{12}, x_{22})$

$$\phi(x_2) \leq \underline{x}'_1 + \epsilon < \bar{x}_1 + \eta_2 + \epsilon = \underline{x}_1 + \epsilon + \eta_2 + \epsilon < x_{01} + \eta_1 + \eta_2 + 2\epsilon < \phi(x_{02}) + 4\epsilon$$

which is the upper semi-continuity of ϕ at x_{02}. □

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