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3. Let $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and let S be that portion of the plane $2x + y + 2z = 6$ that is in the first quadrant (draw a figure). Assuming that S is oriented upward, we would like to verify the Stokes theorem i.e.

$$\oint_C \mathbf{F}.d\mathbf{r} = \int\int_S (curl\mathbf{F}).\mathbf{n}\ dS. \tag{1}$$

S is defined by the equation $z = f(x, y) = 3 - x - \frac{1}{2}y$. So the unit normal vector to S is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\big(- f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}\big) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\big(\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}\big). \tag{2}$$

$$\begin{aligned} curl\mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} \\ &= (1 - 0)\mathbf{i} - (0 - 1)\mathbf{j} + (1 - 0)\mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k}. \end{aligned} \tag{3}$$

Now let R be the projection of S on the $xy-$plane. Taking into account (2) and (3), we obtain for the right-hand side of (1)

$$\begin{aligned} \int\int_S (curl\mathbf{F}).\mathbf{n}\ dS &= \int\int_S (\mathbf{i} + \mathbf{j} + \mathbf{k}).\frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\big(\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}\big)dS \\ &= \int\int_R (\mathbf{i} + \mathbf{j} + \mathbf{k}).\frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\big(\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}\big)\sqrt{1 + f_x^2 + f_y^2}dxdy \\ &= \int\int_R (1 + \frac{1}{2} + 1)dxdy = \frac{5}{2}\int\int_R dxdy = \frac{5}{2}Area(R). \end{aligned} \tag{4}$$

Since R is bounded by the triangle with vertices $(3, 0, 0)$, $(0, 6, 0)$ and $(0, 0, 0)$, we have $Area(R) = (3)(6)/2 = 9$ and hence we get from (4)

$$\int\int_S (curl\mathbf{F}).\mathbf{n}\ dS = \frac{45}{2}. \tag{5}$$

Now to evaluate the left-hand side of (1), notice that $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment joining the points $(3,0,0)$ and $(0,6,0)$, C_2 is the line segment joining the points $(0,6,0)$ and $(0,0,3)$, and where C_3 is the line segment joining the points $(0,0,3)$ and $(3,0,0)$. C_1, C_2 and C_3 have the parameterizations

$$C_1 : \begin{cases} x = 3 - \frac{1}{2}t \\ y = t, & t \in [0,6] \\ z = 0, \end{cases} \qquad C_2 : \begin{cases} x = 0 \\ y = 6 - 2t, & t \in [0,3] \\ z = t, \end{cases}$$

$$\text{and} \qquad C_3 : \begin{cases} x = t \\ y = 0, & t \in [0,3] \\ z = 3 - t. \end{cases}$$

Then we have

$$\begin{aligned} \oint_C \mathbf{F}.d\mathbf{r} = \oint_C zdx + xdy + ydz &= \int_{C_1} zdx + xdy + ydz + \int_{C_2} zdx + xdy + ydz \\ &+ \int_{C_3} zdx + xdy + ydz. \end{aligned} \tag{6}$$

$$\begin{aligned} \int_{C_1} zdx + xdy + ydz &= \int_{C_1} zdx + \int_{C_1} xdy + \int_{C_1} ydz \\ &= \int_0^6 (0)\big(-\frac{1}{2}\big)dt + \int_0^6 \big(3 - \frac{1}{2}t\big)dt + \int_0^6 t(0)dt \\ &= \int_0^6 \big(3 - \frac{1}{2}t\big)dt = [3t - \frac{1}{4}t^2]_0^6 = 18 - 9 = 9. \end{aligned} \tag{7}$$

$$\begin{aligned} \int_{C_2} zdx + xdy + ydz &= \int_{C_2} zdx + \int_{C_2} xdy + \int_{C_2} ydz \\ &= \int_0^3 t(0)dt + \int_0^3 0(-2)dt + \int_0^3 (6 - 2t)dt \\ &= \int_0^3 (6 - 2t)dt = [6t - t^2]_0^3 = 9. \end{aligned} \tag{8}$$

$$
\begin{aligned}
\int_{C_3} zdx + xdy + ydz &= \int_{C_3} zdx + \int_{C_3} xdy + \int_{C_3} ydz \\
&= \int_0^3 (3-t)dt + \int_0^3 t(0)dt + \int_0^3 (0)(-1)dt \\
&= \int_0^3 (3-t)dt = [3t - \frac{1}{2}t^2]_0^3 = 9 - \frac{9}{2} = \frac{9}{2}. \qquad (9)
\end{aligned}
$$

Using (6)-(9), we get

$$\oint_C \mathbf{F}.d\mathbf{r} = 9 + 9 + \frac{9}{2} = \frac{45}{2}. \qquad (10)$$

Finally by comparing (5) and (10), we conclude that (1) is satisfied. □

6. Let $\mathbf{F} = z^2 y \cos(xy)\mathbf{i} + z^2 x(1 + \cos(xy))\mathbf{j} + 2z\sin(xy)\mathbf{k}$ and let S be the portion of the plane $z = 1 - y$ in the first octant that is located within the planes $x = 0$ and $x = 2$ (see the figure). The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. Therefore we have by Stokes' theorem

$$\oint_C \mathbf{F}.d\mathbf{r} = \int\int_S curl(\mathbf{F}).\mathbf{n}dS, \qquad (1)$$

where

$$
\begin{aligned}
curl\mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 y\cos(xy) & z^2 x(1+\cos(xy)) & 2z\sin(xy) \end{vmatrix} \\
&= (2xz\cos(xy) - 2xz(1+\cos(xy)))\mathbf{i} - (2yz\cos(xy) - 2yz\cos(xy))\mathbf{j} \\
&\quad +((z^2(1+\cos(xy)) - z^2 xy\sin(xy)) - (z^2\cos(xy) - z^2 xy\sin(xy)))\mathbf{k} \\
&= -2xz\mathbf{i} + z^2\mathbf{k}. \qquad (2)
\end{aligned}
$$

S is defined by the equation $z = f(x, y) = 1 - y$. So the unit normal vector to S is given by

$$\mathbf{n} = \frac{1}{\sqrt{1+f_x^2+f_y^2}}\big(-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}\big) = \frac{1}{\sqrt{2}}\big(\mathbf{j}+\mathbf{k}\big). \tag{3}$$

Now let R be the projection of S on the $xy-$plane. Then we get by using (1), (2) and (3)

$$\begin{aligned}
\oint_C \mathbf{F}.d\mathbf{r} &= \int\int_S \frac{1}{\sqrt{2}} z^2 dS = \int\int_R \frac{1}{\sqrt{2}}\sqrt{2}(1-y)^2 dxdy \\
&= \int_0^2 \Big(\int_0^1 (1-y)^2 dy\Big) dx = \int_0^2 \Big[-\frac{1}{3}(1-y)^3\Big]_0^1 dx \\
&= \int_0^2 \frac{1}{3} dx = \frac{2}{3}.
\end{aligned}$$

□

14. Let $\mathbf{F} = y\mathbf{i} + (y-x)\mathbf{j} + z^2\mathbf{k}$ and let S be the portion of the sphere $x^2+y^2+(z-4)^2 = 25$ that is above the $xy-$plane. We would like to evaluate the surface integral $\displaystyle\int\int_S curl(\mathbf{F}).\mathbf{n}dS$. The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous everywhere. Therefore we have by Stokes' theorem

$$\int\int_S curl(\mathbf{F}).\mathbf{n}dS = \oint_C \mathbf{F}.d\mathbf{r}, \tag{1}$$

where C is the circle of the $xy-$plane of radius 3(take $z=0$ in the equation of the sphere) centered at the point $(0,0,0)$. C has the parameterization

$$C_1 : \begin{cases} x = 3\cos t \\ y = 3\sin t, \quad t \in [0, 2\pi] \\ z = 0, \end{cases}$$

Then we have from (1)

$$\int\int_S curl(\mathbf{F}).\mathbf{n}dS = \int_C ydx + \int_C (y-x)dy + \int_C z^2dz$$

$$\int_0^{2\pi} 3\sin t(-3\sin t)dt + \int_0^{2\pi} 3(\sin t - \cos t)(3\cos t)dt + \int_0^{2\pi} 0^2(0)dt$$

$$\int_0^{2\pi} (-9\sin^2(t) + 9\sin t\cos t - 9\cos^2(t))dt$$

$$= \int_0^{2\pi} (-9 + 9\sin t\cos t)dt = \big[-9t + \frac{9}{2}\sin^2 t\big]_0^{2\pi} = -18\pi.$$

□

17. We would like to evaluate the line integral $\oint_C z^2e^{x^2}dx + xy^2dy + \tan^{-1} ydz$, where C is the circle $x^2+y^2 = 9$ located on the plane $z = 0$. Let $\mathbf{F}(x, y, z) = z^2e^{x^2}\mathbf{i} + xy^2\mathbf{j} + \tan^{-1} y\mathbf{k}$ and S be the portion of the plane $z = 0$ that is bounded by C. We assume that S is oriented upward and C is oriented accordingly. We are going to use Stokes' theorem

$$\oint_C z^2e^{x^2}dx + xy^2dy + \tan^{-1} ydz = \oint_C \mathbf{F}.d\mathbf{r} = \int\int_S curl(\mathbf{F}).\mathbf{n}\ dS. \qquad (1)$$

$$\begin{aligned} curl\mathbf{F} &= \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2e^{x^2} & xy^2 & \tan^{-1} y \end{vmatrix} \\ &= \frac{1}{1+y^2}\mathbf{i} + 2ze^{x^2}\mathbf{j} + y^2\mathbf{k}. \qquad (2) \end{aligned}$$

S is defined by $z = f(x, y) = 0$. So the unit normal vector to S is given by $\mathbf{n} = \mathbf{k}$. Let R be the projection of S on the $xy-$plane. Then we deduce from (2)

$$\begin{aligned}\int\int_S \mathbf{F}.\mathbf{n}\; dS &= \int\int_S \frac{1}{\sqrt{1+f_x^2+f_y^2}} y^2\; dS \\ &= \int\int_R \frac{1}{\sqrt{1+f_x^2+f_y^2}} y^2 \sqrt{1+f_x^2+f_y^2} dxdy \\ &= \int\int_R y^2 dxdy. \end{aligned} \tag{3}$$

Since R is bounded by the circle centered at the origin and with radius 3, we can use the polar coordinates. We obtain

$$\begin{aligned}\int\int_R y^2 dxdy &= \int_0^3 \Big(\int_0^{2\pi} r^3 \sin^2\theta d\theta \Big) dr \\ &= \int_0^3 \Big(\int_0^{2\pi} \frac{r^3}{2} (1-\cos(2\theta)) \Big) d\theta \Big) \\ &= \int_0^3 \Big[\frac{r^3}{2} (\theta - \frac{1}{2}\sin(2\theta)) \Big]_0^{2\pi} dr \\ &= \int_0^3 \pi r^3 dr = \Big[\frac{\pi}{4} r^4 \Big]_0^3 = \frac{81\pi}{4}. \end{aligned} \tag{4}$$

Taking into account (1), (3) and (4), we get

$$\oint_C z^2 e^{x^2} dx + xy^2 dy + \tan^{-1} y dz = \frac{81\pi}{4}.$$

□