King Fahd University of Petroleum and Minerals

Department of Mathematical Sciences

Dr. A. Lyaghfouri

MATH 301/Term 062/Hw#6(9.12)/

3. We would like to verify Green's theorem in the following situation: $P(x,y) = -y^2$, $Q(x,y) = x^2$ and R is the region of the $xy-$plane bounded by the circle C of center $(0,0)$ and radius 3 i.e.

$$\oint_C P dx + Q dy = \int\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dxdy$$

or

$$\oint_C -y^2 dx + x^2 dy = \int\int_R (2x + 2y) dxdy. \qquad (1)$$

First note that $P(x,y)$ and $Q(x,y)$ are continuous and have partial derivatives continuous on any domain. Moreover we have $\dfrac{\partial P}{\partial y} = -2y$ and $\dfrac{\partial Q}{\partial x} = 2x$
Next the circle C has the parametrization

$$C : \begin{cases} x = 3\cos(t), \\ y = 3\sin(t), \quad t \in [0, 2\pi]. \end{cases}$$

Then we have

$$\begin{aligned}
\int_C -y^2 dx + x^2 dy &= \int_C -y^2 dx + \int_C x^2 dy \\
&= \int_0^{2\pi} -9\sin^2(t)(-3)\sin(t)dt + \int_0^{2\pi} 9\cos^2(t) 3\cos(t)dt \\
&= \int_0^{2\pi} 27(1-\cos^2(t))\sin(t)dt + \int_0^{2\pi} 27(1-\sin^2(t))\cos(t)dt \\
&= \int_0^{2\pi} (27\sin(t) - 27\cos^2(t)\sin(t))dt + \int_0^{2\pi} (27\cos(t) - 27\sin^2(t)\cos(t)dt \\
&= [-27\cos(t) + 9\cos^3(t)]_0^{2\pi} + [27\sin(t) - 9\sin^3(t)]_0^{2\pi} = 0. \qquad (2)
\end{aligned}$$

Using the polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we get

$$\begin{aligned}
\int\int_R (2x+2y)dxdy &= \int_0^3\int_0^{2\pi} 2r^2(\cos(\theta)+\sin(\theta))drd\theta \\
&= \int_0^3 2r^2\Big(\int_0^{2\pi}(\cos(\theta)+\sin(\theta))d\theta\Big)dr \\
&= \int_0^3 2r^2\Big[-\sin(\theta)+\cos(\theta)\Big]_0^{2\pi}dr \\
&= \int_0^3 2r^2(0)dr = 0. \qquad (3)
\end{aligned}$$

Using (2) and (3), we conclude that (1) is true.

□

6. We would like to evaluate the line integral $\oint_C (x+y^2)dx+(2x^2-y)dy$, where C is the boundary of the region determined by the graphs of $y=x^2$ and $y=4$ (draw a figure). Let $P(x,y)=x+y^2$ and $Q(x,y)=2x^2-y$. The functions P and Q are continuous and have partial derivatives continuous on any domain and moreover we have $\dfrac{\partial P}{\partial y}=2y$ and $\dfrac{\partial Q}{\partial x}=4x$. Using Green's theorem we have

$$\oint_C Pdx+Qdy = \int\int_R\Big(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\Big)dxdy$$

or

$$\oint_C (x+y^2)dx+(2x^2-y)dy = \int\int_R (4x-2y)dxdy. \qquad (1)$$

We will evaluate the second integral in (1).

$$
\begin{aligned}
\int\int_R (4x-2y)dxdy &= \int_{-2}^{2}\Big(\int_{x^2}^{4}(4x-2y)dy\Big)dx \\
&= \int_{-2}^{2}\Big[4xy-y^2\Big]_{x^2}^{4}dx \\
&= \int_{-2}^{2}\big[(16x-16)-(4x^3-x^4)\big]dx \\
&= \int_{-2}^{2}(x^4-4x^3+16x-16)dx = \Big[\frac{1}{5}x^5-x^4+8x^2-16x\Big]_{-2}^{2} \\
&= \frac{2}{5}2^5-64 = 64(\frac{1}{5}-1) = -\frac{256}{5}. \qquad (2)
\end{aligned}
$$

Using (1) and (2), we obtain $\oint_C (x+y^2)dx+(2x^2-y)dy = -\frac{256}{5}$.

□

18. We would like to prove the following result:

$$
\frac{1}{2}\oint_C -ydx+xdy = area(R), \qquad (1)
$$

where R is the region of the $xy-$plane bounded by a piecewise smooth simple closed curve C.
Let $P(x,y) = -y$ and $Q(x,y) = x$. Note that $P(x,y)$ and $Q(x,y)$ are continuous and have partial derivatives continuous on any domain. Moreover we have $\frac{\partial P}{\partial y} = -1$ and $\frac{\partial Q}{\partial x} = 1$. Using Green's theorem we have

$$
\oint_C Pdx+Qdy = \int\int_R \Big(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\Big)dxdy
$$

or

$$
\oint_C -ydx+xdy = \int\int_R 2dxdy = 2\int\int_R dxdy = 2area(R), \qquad (2)
$$

which leads to (1) after division by 2.

□

25. We would like to evaluate the line integral $\displaystyle\int_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}$, where C is the ellipse $x^2 + 4y^2 = 4$. We will use Green's theorem.
Let R be the region of the $xy-$plane bounded by C and the circle C' : $4x^2 + 4y^2 = 1$ of center $(0,0)$ and radius $1/2$.
Let $P(x,y) = \frac{-y^3}{(x^2+y^2)^2}$ and $Q(x,y) = \frac{xy^2}{(x^2+y^2)^2}$. Thsese functions are continuous and have partial derivatives continuous on any domain not containing the origin. Moreover we have $\displaystyle\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[-y^3(x^2+y^2)^{-2}] = -3y^2(x^2+y^2)^{-2}+4y^4(x^2+y^2)^{-3} = (y^4-3x^2y^2)(x^2+y^2)^{-3},$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[xy^2(x^2+y^2)^{-2}] = y^2(x^2+y^2)^{-2}-4x^2y^2(x^2+y^2)^{-3} = (y^4-3x^2y^2)(x^2+y^2)^{-3}.$$

Using Green's theorem we have

$$\oint_{C\cup C'} Pdx + Qdy = \int\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy = 0.$$

Taking into account that orientations of C and C' in the left hand-side of the previous formula are respectively counterclockwise and clockwise, we get:

$$\int_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} = \int_{C'} \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}, \tag{1}$$

where now the orientations of C and C' are both counterclockwise.
It is clear that it is easier to evaluate the second integral in (1) which we will do by using the parametrization of C':

$$C : \begin{cases} x = \frac{1}{2}\cos(t), \\ y = \frac{1}{2}\sin(t), \quad t \in [0, 2\pi]. \end{cases}$$

Then we have

$$\begin{aligned}
\int_{C'} \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} &= \int_{C'} \frac{-y^3dx}{(x^2+y^2)^2} + \int_{C'} \frac{xy^2dy}{(x^2+y^2)^2} \\
&= \int_0^{2\pi} \frac{\frac{1}{16}\sin^4(t)}{\frac{1}{16}} dt + \int_0^{2\pi} \frac{\frac{1}{16}\sin^2(t)\cos^2(t)}{\frac{1}{16}} dt \\
&= \int_0^{2\pi} \sin^4(t)dt + \int_0^{2\pi} \sin^2(t)\cos^2(t)dt = \int_0^{2\pi} \sin^2(t)(\sin^2(t)+\cos^2(t))dt \\
&= \int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} \frac{1}{2}(1-\cos(2t))dt = \frac{1}{2}\Big[t - \frac{1}{2}\sin(2t)\Big]_0^{2\pi} = \pi. \qquad (2)
\end{aligned}$$

Taking into account (1) and (2), we obtain

$$\int_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} = \pi.$$

□