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6. The line integral $\displaystyle\int_{(1,0)}^{(3,4)} \frac{xdx + ydy}{\sqrt{x^2+y^2}}$ is independent of path. Indeed it is of the form $\displaystyle\int_{(1,0)}^{(3,4)} Pdx + Qdy$, with $P(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ and $Q(x,y) = \frac{y}{\sqrt{x^2+y^2}}$ which are continuous and have partial derivatives continuous on any domain not containing the origin. Moreover we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[x(x^2+y^2)^{-1/2}] = x(-1/2)2y(x^2+y^2)^{-3/2} = -xy(x^2+y^2)^{-3/2},$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[y(x^2+y^2)^{-1/2}] = y(-1/2)2x(x^2+y^2)^{-3/2} = -xy(x^2+y^2)^{-3/2}.$$

a) Since the integral is independent of path, there exists a function ϕ such that $d\phi = Pdx + Qdy$ i.e.

$$\frac{\partial \phi}{\partial x} = P(x,y) = \frac{x}{\sqrt{x^2+y^2}} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = Q(x,y) = \frac{y}{\sqrt{x^2+y^2}}. \tag{2}$$

Integrating (1), we get

$$\phi(x,y) = \int \frac{xdx}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} + g(y). \tag{3}$$

Using (2) and (3), we get

$$\frac{y}{\sqrt{x^2+y^2}} + g'(y) = \frac{y}{\sqrt{x^2+y^2}} \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = C. \tag{4}$$

Combining (3) and (4), we obtain

$$\phi(x,y) = \sqrt{x^2+y^2} + C.$$

Hence

$$\int_{(1,0)}^{(3,4)} \frac{xdx+ydy}{\sqrt{x^2+y^2}} = \phi(3,4) - \phi(1,0) = \sqrt{3^2+4^2} - \sqrt{1^2+0^2} = \sqrt{25} - 1 = 4.$$

b) We consider the piecewise smooth curve $C = C_1 \cup C_2$ joining the points $(1,0)$ and $(3,4)$, where C_1 is the horizontal line segment joining the points $(1,0)$ and $(3,0)$, and where C_2 is the vertical line segment joining the points $(3,0)$ and $(3,4)$. C_1 and C_2 have the parameterizations

$$C_1 : \begin{cases} x = t, \\ y = 0, \quad t \in [1,3] \end{cases} \qquad \text{and} \qquad C_2 : \begin{cases} x = 3, \\ y = t, \quad t \in [0,4]. \end{cases}$$

Then we have

$$\int_{(1,0)}^{(3,4)} \frac{xdx+ydy}{\sqrt{x^2+y^2}} = \int_C \frac{xdx+ydy}{\sqrt{x^2+y^2}} = \int_{C_1} \frac{xdx+ydy}{\sqrt{x^2+y^2}} + \int_{C_2} \frac{xdx+ydy}{\sqrt{x^2+y^2}}. \quad (5)$$

$$\begin{aligned} \int_{C_1} \frac{xdx+ydy}{\sqrt{x^2+y^2}} &= \int_{C_1} \frac{x}{\sqrt{x^2+y^2}}dx + \int_{C_1} \frac{y}{\sqrt{x^2+y^2}}dy \\ &= \int_1^3 \frac{t}{\sqrt{t^2+0^2}}dt + \int_1^3 0dt \\ &= \int_1^3 dt = [t]_1^3 = [3-1] = 2. \end{aligned} \quad (6)$$

$$\begin{aligned} \int_{C_2} \frac{xdx+ydy}{\sqrt{x^2+y^2}} &= \int_{C_2} \frac{x}{\sqrt{x^2+y^2}}dx + \int_{C_2} \frac{y}{\sqrt{x^2+y^2}}dy \\ &= \int_0^4 \frac{3}{\sqrt{3^2+t^2}}(0)dt + \int_0^4 \frac{t}{\sqrt{3^2+t^2}}dt = \int_0^4 \frac{t}{\sqrt{9+t^2}}dt \\ &= [\sqrt{9+t^2}]_0^4 = [\sqrt{25} - \sqrt{9}] = 5 - 3 = 2. \end{aligned} \quad (7)$$

Using (5), (6) and (7), we get

$$\int_{(1,0)}^{(3,4)} \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = 2 + 2 = 4.$$

$\square$

15. Let $\mathbf{F}(x, y) = (x^3 + y)\mathbf{i} + (x + y^3)\mathbf{j} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field. Since the functions P and Q are continuous and have partial derivatives continuous on any domain and moreover we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = 1$, $\mathbf{F}$ is a gradient field i.e. there exists a function ϕ such that $\nabla\phi(x, y) = \mathbf{F}(x, y)$ i.e.

$$\frac{\partial \phi}{\partial x} = P(x, y) = x^3 + y \tag{1}$$

$$\frac{\partial \phi}{\partial y} = Q(x, y) = x + y^3. \tag{2}$$

Integrating (1), we get

$$\phi(x, y) = \int (x^3 + y)dx = \frac{1}{4}x^4 + xy + g(y). \tag{3}$$

Using (2) and (3), we get

$$x + g'(y) = x + y^3 \;\Rightarrow\; g'(y) = y^3 \;\Rightarrow\; g(y) = \frac{1}{4}y^4 + C. \tag{4}$$

Combining (3) and (4), we obtain

$$\phi(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + xy + C.$$

$\square$

18. Let $\mathbf{F}(x, y) = (2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}$. We would like to evaluate the work W done by the force $\mathbf{F}$ along the part of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ located above the x-axis.

We remark that the components P and Q of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain and moreover we have

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = -e^{-y}$. Hence $\mathbf{F}$ is a gradient field i.e. there exists a function ϕ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$ i.e.

$$\frac{\partial \phi}{\partial x} = P(x,y) = 2x + e^{-y} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = Q(x,y) = 4y - xe^{-y}. \tag{2}$$

Integrating (1), we get

$$\phi(x,y) = \int (2x + e^{-y})dx = x^2 + xe^{-y} + g(y). \tag{3}$$

Using (2) and (3), we get

$$-xe^{-y} + g'(y) = 4y - xe^{-y} \;\Rightarrow\; g'(y) = 4y \;\Rightarrow\; g(y) = 2y^2 + C. \tag{4}$$

Combining (3) and (4), we obtain

$$\phi(x,y) = x^2 + xe^{-y} + 2y^2 + C.$$

Hence we have

$$\begin{aligned} W = \int_C \mathbf{F}.d\mathbf{r} &= \int_C \big((2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}\big).(dx\mathbf{i} + dy\mathbf{j}) \\ &= \phi(-2,0) - \phi(2,0) = (4 - 2e^0 + 0) - (4 + 2e^0 + 0) = -4. \end{aligned}$$

□

24. The line integral $\displaystyle\int_{(-2,3,1)}^{(0,0,0)} 2xzdx + 2yzdy + (x^2 + y^2)dz$ is independent of path. Indeed it is of the form $\displaystyle\int_{(-2,3,1)}^{(0,0,0)} Pdx + Qdy + Rdz$, with $P(x,y,z) = 2xz$, $Q(x,y,z) = 2yz$ and $R(x,y,z) = x^2 + y^2$ which are continuous and have partial derivatives continuous on any domain. Moreover we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 2x \;\text{ and }\; \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 2y.$$

Since the integral is independent of path, there exists a function ϕ such that $d\phi = Pdx + Qdy + Rdz$ i.e.

$$\frac{\partial \phi}{\partial x} = P(x,y,z) = 2xz \tag{1}$$
$$\frac{\partial \phi}{\partial y} = Q(x,y,z) = 2yz \tag{2}$$
$$\frac{\partial \phi}{\partial z} = R(x,y,z) = x^2 + y^2. \tag{3}$$

Integrating (1), we get

$$\phi(x,y,z) = \int 2xzdx = x^2z + g(y,z). \tag{4}$$

Using (2) and (4), we get

$$\frac{\partial g}{\partial y} = 2yz \;\Rightarrow\; g(y,z) = y^2z + h(z) \;\Rightarrow\; \phi(x,y,z) = x^2z + y^2z + h(z). \tag{5}$$

Using (3) and (5), we get

$$x^2 + y^2 + h'(z) = x^2 + y^2 \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = C. \tag{6}$$

Combining (5) and (6), we obtain

$$\phi(x,y,z) = x^2z + y^2z + C.$$

Hence

$$\int_{(-2,3,1)}^{(0,0,0)} 2xzdx+2yzdy+(x^2+y^2)dz = \phi(0,0,0)-\phi(-2,3,1) = 0-(-2)^2-3^2 = -4-9 = -13.$$

□

28. Let $\mathbf{F}(x,y,z) = 8xy^3z\mathbf{i} + 12x^2y^2z\mathbf{j} + 4x^2y^3\mathbf{k}$. We would like to evaluate the works W_1 and W_2 done by the force $\mathbf{F}$ acting along the helix $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + t\mathbf{k}$ from $(2,0,0)$ to $(1,\sqrt{3},\pi/3)$ and from $(2,0,0)$ to $(0,2,\pi/2)$ respectively.

We remark that the components P, Q and R of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain and satisfy

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 24xy^2z, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 8xy^3 \text{ and } \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 12x^2y^2.$$

We deduce that the integral is independent of path. Moreover there exists a function ϕ such that $\nabla\phi = \mathbf{F}$ or equivalently $d\phi = Pdx + Qdy + Rdz$ i.e.

$$\frac{\partial \phi}{\partial x} = P(x,y,z) = 8xy^3z \tag{1}$$
$$\frac{\partial \phi}{\partial y} = Q(x,y,z) = 12x^2y^2z \tag{2}$$
$$\frac{\partial \phi}{\partial z} = R(x,y,z) = 4x^2y^3. \tag{3}$$

Integrating (1), we get

$$\phi(x,y,z) = \int 8xy^3zdx = 4x^2y^3z + g(y,z). \tag{4}$$

Using (2) and (4), we get

$$12x^2y^2z + \frac{\partial g}{\partial y} = 12x^2y^2z \quad \Rightarrow \quad \frac{\partial g}{\partial y} = 0 \ \Rightarrow \ g(y,z) = h(z)$$
$$\Rightarrow \quad \phi(x,y,z) = 4x^2y^3z + h(z). \tag{5}$$

Using (3) and (5), we get

$$4x^2y^3 + h'(z) = 4x^2y^3 \ \Rightarrow \ h'(z) = 0 \ \Rightarrow \ h(z) = C. \tag{6}$$

Combining (5) and (6), we obtain

$$\phi(x,y,z) = 4x^2y^3z + C.$$

Hence

$$W_1 = \int_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} \mathbf{F}.d\mathbf{r} = \phi(1,\sqrt{3},\pi/3) - \phi(2,0,0) = 4(1)^2(\sqrt{3})^3(\pi/3) - 0 = 4\pi\sqrt{3}$$

and

$$W_2 = \int_{(2,0,0)}^{(0,2,\pi/2)} \mathbf{F}.d\mathbf{r} = \phi(0,2,\pi/2) - \phi(2,0,0) = 0 - 0 = 0.$$

□