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MATH 301/Term 062/Hw#23(13.4)/

2. We would like to solve the following boundary-value problem

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{for all} \quad 0 < x < L, t > 0 \tag{1}$$
$$u(0,t) = 0, \ t > 0 \tag{2}$$
$$u(L,t) = 0, \ t > 0 \tag{3}$$
$$u(x,0) = 0, \ 0 < x < L \tag{4}$$
$$\frac{\partial u}{\partial t}(x,0) = x(L-x), \ 0 < x < L. \tag{5}$$

Let us find all product solutions of the boundary-value problem (1), (2), (3) and (4) Indeed let $u(x,y) = X(x)T(t)$ be such a product solution. Then we have

$$a^2 X''(x)T(t) = X(x)T''(t) \quad \text{for all} \quad x, t. \tag{6}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (6)

$$a^2\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = constant\ k \quad \text{for all} \quad x, t. \tag{7}$$

We deduce from (7) that

$$X''(x) = \frac{k}{a^2}X(x) \tag{8}$$
$$T''(t) = kT(t). \tag{9}$$

The solutions of (8) and (9) depend on the sign of k, i.e. we have

$$X(x) = c_1 x + c_2 \text{ if } k = 0 \tag{10}$$
$$X(x) = c_1 \cosh\left(\frac{\sqrt{k}}{a}x\right) + c_2 \sinh\left(\frac{\sqrt{k}}{a}x\right) \text{ if } k > 0 \tag{11}$$
$$X(x) = c_1 \cos\left(\frac{\sqrt{-k}}{a}x\right) + c_2 \sin\left(\frac{\sqrt{-k}}{a}x\right) \text{ if } k < 0. \tag{12}$$

$$\begin{aligned}
T(t) &= c_3 t + c_4 \text{ if } k = 0 && (13)\\
T(t) &= c_3 \cosh(\sqrt{k}t) + c_4 \sinh(\sqrt{k}t) \text{ if } k > 0 && (14)\\
T(t) &= c_3 \cos(\sqrt{-k}t) + c_4 \sin(\sqrt{-k}t) \text{ if } k < 0. && (15)
\end{aligned}$$

We discuss three cases:

<u>*Case 1*: $k = 0$</u>

In this case we have by (10) and (13) $u(x,t) = X(x)T(t) = (c_1 x + c_2)(c_3 t + c_4)$. Using (2) and (3), we get

$$\begin{cases} c_2 T(t) = 0 & \forall t > 0 \\ (c_1 L + c_2) T(t) = 0 & \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_2 T(t) = 0 & \forall t > 0 \\ c_1 T(t) = 0 & \forall t > 0 \end{cases} \Rightarrow u(x,t) \equiv 0.$$

<u>*Case 2*: $k = \lambda^2 > 0$</u>

In this case we have by (11) and (14) that

$$u(x,t) = X(x)T(t) = \left(c_1 \cosh\left(\frac{\lambda}{a}x\right) + c_2 \sinh\left(\frac{\lambda}{a}x\right)\right)(c_3 \cosh(\lambda t) + c_4 \sinh(\lambda t)).$$

Using (2) and (3), we get

$$\begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ \left(c_1 \cosh\left(\frac{\lambda}{a}L\right) + c_2 \sinh\left(\frac{\lambda}{a}L\right)\right) T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ c_2 T(t) = 0 \quad \forall t > 0. \end{cases}$$

Hence $u(x,t) \equiv 0$.

<u>*Case 3*: $k = -\lambda^2 < 0$</u>

In this case we have by (12) and (15) that

$$u(x,t) = X(x)T(t) = \left(c_1 \cos\left(\frac{\lambda}{a}x\right) + c_2 \sin\left(\frac{\lambda}{a}x\right)\right)(c_3 \cos(\lambda t) + c_4 \sin(\lambda t)).$$

Using (2) and (3), we get

$$\begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ \left(c_1 \cos\left(\frac{\lambda}{a}L\right) + c_2 \sin\left(\frac{\lambda}{a}L\right)\right) T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ c_2 \sin\left(\frac{\lambda}{a}L\right) T(t) = 0 \quad \forall t > 0 \end{cases}$$

If $\sin\left(\frac{\lambda}{a}L\right) \neq 0$, then $c_2 T(t) = 0 \ \forall t > 0$ and then $u(x,t) \equiv 0$.

If $\sin\left(\frac{\lambda}{a}L\right) = 0$, we obtain $\frac{\lambda}{a}L = n\pi$, i.e. $\lambda = \frac{n\pi a}{L}$, $n = 1, 2, ...$ and

$$u(x,t) = c_2 \sin\left(\frac{\lambda}{a}x\right)(c_3\cos(\lambda t) + c_4\sin(\lambda t)).$$

Using now the boundary condition (4), we get

$$u(x,0) = c_2 c_3 \sin\left(\frac{\lambda}{a}x\right) = 0,\ \forall 0 < x < L.$$

This leads to

$$u(x,t) = c_2 c_4 \sin\left(\frac{\lambda}{a}x\right)\sin(\lambda t).$$

Therefore all product solutions of (1), (2), (3) and (4) are given in this case by

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right) \tag{16}$$

is also a solution of (1), (2), (3) and (4).
Now it is enough to find the coefficients B_n such that the function given in (16) is also a solution of (5) i.e.

$$x(L-x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L}\sin\left(\frac{n\pi}{L}x\right), \quad 0 < x < L. \tag{17}$$

It is clear that (17) is the half-range expansion of the function $x(L-x)$ in a sine series. it follows that

$$B_n \frac{n\pi a}{L} = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

and

$$\begin{aligned} B_n &= \frac{2}{n\pi a}\int_0^L x(L-x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= \frac{2L}{n\pi a}\int_0^L x\sin\left(\frac{n\pi}{L}x\right)dx - \frac{2}{n\pi a}\int_0^L x^2\sin\left(\frac{n\pi}{L}x\right)dx. \end{aligned} \tag{18}$$

Integrating by parts, we get

$$
\begin{aligned}
\int_0^L x\sin\left(\frac{n\pi}{L}x\right)dx &= \Big[-x\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big]_0^L - \int_0^L -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)dx \\
&= -\frac{L^2}{n\pi}\cos(n\pi) + 0 + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{(-1)^{n+1}L^2}{n\pi} + \frac{L}{n\pi}\Big[\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big]_0^L \\
&= \frac{(-1)^{n+1}L^2}{n\pi} + \frac{L^2}{n^2\pi^2}(\sin(n\pi) - 0) \\
&= \frac{(-1)^{n+1}L^2}{n\pi}. \qquad (19)
\end{aligned}
$$

Integrating by parts twice, we get

$$
\begin{aligned}
\int_0^L x^2\sin\left(\frac{n\pi}{L}x\right)dx &= \Big[-x^2\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big]_0^L - \int_0^L -2x\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)dx \\
&= -\frac{L^3}{n\pi}\cos(n\pi) + \frac{2L}{n\pi}\int_0^L x\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + \frac{2L}{n\pi}\left(\Big[x\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big]_0^L - \int_0^L \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)dx\right) \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^2}{n^2\pi^2}(L\sin(n\pi) - 0) - 2\frac{L^2}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^3}{n^3\pi^3}\Big[\cos\left(\frac{n\pi}{L}x\right)\Big]_0^L \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^3}{n^3\pi^3}((-1)^n - 1). \qquad (20)
\end{aligned}
$$

Taking into account (18), (19) and (20), we deduce that

$$
\begin{aligned}
B_n &= \frac{2L}{n\pi a}\frac{(-1)^{n+1}L^2}{n\pi} - \frac{2}{n\pi a}\left(\frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^3}{n^3\pi^3}((-1)^n - 1)\right) \\
&= \frac{4L^3}{an^4\pi^4}(1 - (-1)^n). \qquad (21)
\end{aligned}
$$

Hence we obtain from (16) and (21) the solution of our BVP

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{4L^3}{an^4\pi^4}(1-(-1)^n)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right) \\
&= \sum_{n=0}^{\infty} \frac{8L^3}{a(2n+1)^4\pi^4}\sin\left(\frac{(2n+1)\pi}{L}x\right)\sin\left(\frac{(2n+1)\pi a}{L}x\right).
\end{aligned}
$$

□

4. We would like to solve the following boundary-value problem

$$
\begin{aligned}
& a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{for all} \quad 0 < x < \pi, t > 0 && (1) \\
& u(0,t) = 0, \ t > 0 && (2) \\
& u(\pi,t) = 0, \ t > 0 && (3) \\
& \frac{\partial u}{\partial t}(x,0) = 0, \ 0 < x < \pi && (4) \\
& u(x,0) = \frac{1}{6}x(\pi^2 - x^2), \ 0 < x < \pi. && (5)
\end{aligned}
$$

Let us find all product solutions of the boundary-value problem (1), (2), (3) and (4). Indeed let $u(x,y) = X(x)T(t)$ be such a product solution. Then we have

$$
a^2X''(x)T(t) = X(x)T''(t) \quad \text{for all} \quad x,t. \qquad (6)
$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (6)

$$
a^2\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \textit{constant } k \quad \text{for all} \quad x,t. \qquad (7)
$$

We deduce from (7) that

$$
\begin{aligned}
X''(x) &= \frac{k}{a^2}X(x) && (8) \\
T''(t) &= kT(t). && (9)
\end{aligned}
$$

The solutions of (8) and (9) depend on the sign of k, i.e. we have

$$\begin{aligned} X(x) &= c_1 x + c_2 \text{ if } k = 0 & (10)\\ X(x) &= c_1 \cosh\left(\frac{\sqrt{k}}{a}x\right) + c_2 \sinh\left(\frac{\sqrt{k}}{a}x\right) \text{ if } k > 0 & (11)\\ X(x) &= c_1 \cos\left(\frac{\sqrt{-k}}{a}x\right) + c_2 \sin\left(\frac{\sqrt{-k}}{a}x\right) \text{ if } k < 0. & (12) \end{aligned}$$

$$\begin{aligned} T(t) &= c_3 t + c_4 \text{ if } k = 0 & (13)\\ T(t) &= c_3 \cosh(\sqrt{k}t) + c_4 \sinh(\sqrt{k}t) \text{ if } k > 0 & (14)\\ T(t) &= c_3 \cos(\sqrt{-k}t) + c_4 \sin(\sqrt{-k}t) \text{ if } k < 0. & (15) \end{aligned}$$

We discuss three cases:

Case 1: $k = 0$

In this case we have by (10) and (13) $u(x,t) = X(x)T(t) = (c_1 x + c_2)(c_3 t + c_4)$. Using (2) and (3), we get

$$\begin{cases} c_2 T(t) = 0 & \forall t > 0 \\ (c_1\pi + c_2)T(t) = 0 & \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_2 T(t) = 0 & \forall t > 0 \\ c_1 T(t) = 0 & \forall t > 0 \end{cases} \Rightarrow u(x,t) \equiv 0.$$

Case 2: $k = \lambda^2 > 0$

In this case we have by (11) and (14) that

$$u(x,t) = X(x)T(t) = \left(c_1 \cosh\left(\frac{\lambda}{a}x\right) + c_2 \sinh\left(\frac{\lambda}{a}x\right)\right)(c_3 \cosh(\lambda t) + c_4 \sinh(\lambda t)).$$

Using (2) and (3), we get

$$\begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ \left(c_1 \cosh\left(\frac{\lambda}{a}\pi\right) + c_2 \sinh\left(\frac{\lambda}{a}\pi\right)\right)T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_1 T(t) = 0 & \forall t > 0 \\ c_2 T(t) = 0 & \forall t > 0. \end{cases}$$

Hence $u(x,t) \equiv 0$.

Case 3: $k = -\lambda^2 < 0$

In this case we have by (12) and (15) that

$$u(x,t) = X(x)T(t) = \left(c_1 \cos\left(\frac{\lambda}{a}x\right) + c_2 \sin\left(\frac{\lambda}{a}x\right)\right)(c_3 \cos(\lambda t) + c_4 \sin(\lambda t)).$$

Using (2) and (3), we get

$$\begin{cases} c_1 T(t) = 0 \;\; \forall t > 0 \\ \left(c_1 \cos\left(\frac{\lambda}{a}\pi\right) + c_2 \sin\left(\frac{\lambda}{a}\pi\right)\right) T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_1 T(t) = 0 \;\; \forall t > 0 \\ c_2 \sin\left(\frac{\lambda}{a}\pi\right) T(t) = 0 \quad \forall t > 0 \end{cases}$$

If $\sin\left(\frac{\lambda}{a}\pi\right) \neq 0$, then $T(t) = 0 \; \forall t > 0$ and then $u(x,t) \equiv 0$.
If $\sin\left(\frac{\lambda}{a}\pi\right) = 0$, we obtain $\frac{\lambda}{a}\pi = n\pi$, i.e. $\lambda = na$, $n = 1, 2, ...$ and

$$u(x,t) = c_2 \sin\left(\frac{\lambda}{a}x\right)(c_3 \cos(\lambda t) + c_4 \sin(\lambda t)) \tag{16}$$

$$\frac{\partial u}{\partial t}(x,t) = c_2 \lambda \sin\left(\frac{\lambda}{a}x\right)(-c_3 \sin(\lambda t) + c_4 \cos(\lambda t)). \tag{17}$$

Using now the boundary condition (4) and (17), we get

$$0 = \lambda c_2 c_4 \sin\left(\frac{\lambda}{a}x\right), \;\; \forall 0 < x < \pi.$$

This leads by (16) to

$$u(x,t) = c_2 c_3 \sin\left(\frac{\lambda}{a}x\right) \cos(\lambda t).$$

Therefore all product solutions of (1), (2), (3) and (4) are given in this case by

$$u_n(x,t) = B_n \sin(nx) \cos(nat), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) \cos(nat) \tag{18}$$

is also a solution of (1), (2), (3) and (4).
Now it is enough to find the coefficients B_n such that the function given in (18) is also a solution of (5) i.e.

$$\frac{1}{6}x(\pi^2 - x^2) = \sum_{n=1}^{\infty} B_n \sin(nx), \quad 0 < x < \pi. \tag{19}$$

It is clear that (19) is the half-range expansion of the function $\frac{1}{6}x(\pi^2 - x^2)$ in a sine series. it follows that

$$
\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{6}x(\pi^2 - x^2)\sin(nx)dx \\
&= \frac{\pi}{3}\int_0^\pi x\sin(nx)dx - \frac{1}{3\pi}\int_0^\pi x^3\sin(nx)dx. \qquad (20)
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
\int_0^\pi x\sin(nx)dx &= \Big[-x\frac{1}{n}\cos(nx)\Big]_0^\pi - \int_0^\pi -\frac{1}{n}\cos(nx)dx \\
&= -\frac{\pi}{n}\cos(n\pi) + 0 + \frac{1}{n}\int_0^\pi \cos(nx)dx \\
&= \frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\int_0^\pi \cos(nx)dx \\
&= \frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\Big[\frac{1}{n}\sin(nx)\Big]_0^\pi \\
&= \frac{(-1)^{n+1}\pi}{n} + \frac{1}{n^2}(\sin(n\pi) - 0) \\
&= \frac{(-1)^{n+1}\pi}{n}. \qquad (21)
\end{aligned}
$$

Integrating by parts twice, we get

$$
\begin{aligned}
\int_0^\pi x^3\sin(nx)dx &= \Big[-x^3\frac{1}{n}\cos(nx)\Big]_0^\pi - \int_0^\pi -3x^2\frac{1}{n}\cos(nx)dx \\
&= -\frac{\pi^3}{n}\cos(n\pi) + \frac{3}{n}\int_0^\pi x^2\cos(nx)dx \\
&= \frac{(-1)^{n+1}\pi^3}{n} + \frac{3}{n}\Big(\Big[x^2\frac{1}{n}\sin(nx)\Big]_0^\pi - \int_0^\pi 2x\frac{1}{n}\sin(nx)dx\Big) \\
&= \frac{(-1)^{n+1}\pi^3}{n} + \frac{3}{n^2}(\pi^2\sin(n\pi) - 0) - \frac{6}{n^2}\int_0^\pi x\sin(nx)dx \\
&= \frac{(-1)^{n+1}\pi^3}{n} - \frac{6}{n^2}\frac{(-1)^{n+1}\pi}{n} \\
&= \frac{(-1)^{n+1}\pi^3}{n} - \frac{6(-1)^{n+1}\pi}{n^3}. \qquad (22)
\end{aligned}
$$

Taking into account (20), (21) and (22), we deduce that

$$
B_n = \frac{\pi}{3}\frac{(-1)^{n+1}\pi}{n} - \frac{1}{3\pi}\Big(\frac{(-1)^{n+1}\pi^3}{n} - \frac{6(-1)^{n+1}\pi}{n^3}\Big) = \frac{2(-1)^{n+1}}{n^3}. \qquad (23)
$$

Hence we obtain from (18) and (23) the solution of our BVP

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^3} \sin(nx)\cos(n\pi a t).$$

□

6. We would like to solve the following boundary-value problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{for all} \quad 0 < x < 1, t > 0 \tag{1}$$

$$u(0,t) = 0, \; t > 0 \tag{2}$$

$$u(1,t) = 0, \; t > 0 \tag{3}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \; 0 < x < 1 \tag{4}$$

$$u(x,0) = 10^{-2} \sin(3\pi x), \; 0 < x < 1. \tag{5}$$

Let us find all product solutions of the boundary-value problem (1), (2), (3) and (4). Indeed let $u(x,y) = X(x)T(t)$ be such a product solution. Then we have

$$a^2 X''(x)T(t) = X(x)T''(t) \quad \text{for all} \quad x, t. \tag{6}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (6)

$$a^2 \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \textit{constant } k \quad \text{for all} \quad x, t. \tag{7}$$

We deduce from (7) that

$$X''(x) = \frac{k}{a^2} X(x) \tag{8}$$

$$T''(t) = kT(t). \tag{9}$$

The solutions of (8) and (9) depend on the sign of k, i.e. we have

$$X(x) = c_1 x + c_2 \text{ if } k = 0 \tag{10}$$

$$X(x) = c_1 \cosh\left(\frac{\sqrt{k}}{a} x\right) + c_2 \sinh\left(\frac{\sqrt{k}}{a} x\right) \text{ if } k > 0 \tag{11}$$

$$X(x) = c_1 \cos\left(\frac{\sqrt{-k}}{a} x\right) + c_2 \sin\left(\frac{\sqrt{-k}}{a} x\right) \text{ if } k < 0. \tag{12}$$

$$
\begin{aligned}
T(t) &= c_3 t + c_4 \text{ if } k = 0 && (13)\\
T(t) &= c_3 \cosh(\sqrt{k}t) + c_4 \sinh(\sqrt{k}t) \text{ if } k > 0 && (14)\\
T(t) &= c_3 \cos(\sqrt{-k}t) + c_4 \sin(\sqrt{-k}t) \text{ if } k < 0. && (15)
\end{aligned}
$$

We discuss three cases:

Case 1: $k = 0$

In this case we have by (10) and (13) $u(x,t) = X(x)T(t) = (c_1 x + c_2)(c_3 t + c_4)$. Using (2) and (3), we get

$$
\begin{cases} c_2 T(t) = 0 & \forall t > 0 \\ (c_1 + c_2)T(t) = 0 & \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_2 T(t) = 0 & \forall t > 0 \\ c_1 T(t) = 0 & \forall t > 0 \end{cases} \Rightarrow u(x,t) \equiv 0.
$$

Case 2: $k = \lambda^2 > 0$

In this case we have by (11) and (14) that

$$
u(x,t) = X(x)T(t) = \left(c_1 \cosh\left(\frac{\lambda}{a}x\right) + c_2 \sinh\left(\frac{\lambda}{a}x\right)\right)(c_3 \cosh(\lambda t) + c_4 \sinh(\lambda t)).
$$

Using (2) and (3), we get

$$
\begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ \left(c_1 \cosh\left(\frac{\lambda}{a}\right) + c_2 \sinh\left(\frac{\lambda}{a}\right)\right)T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ c_2 T(t) = 0 \quad \forall t > 0. \end{cases}
$$

Hence $u(x,t) \equiv 0$.

Case 3: $k = -\lambda^2 < 0$

In this case we have by (12) and (15) that

$$
u(x,t) = X(x)T(t) = \left(c_1 \cos\left(\frac{\lambda}{a}x\right) + c_2 \sin\left(\frac{\lambda}{a}x\right)\right)(c_3 \cos(\lambda t) + c_4 \sin(\lambda t)).
$$

Using (2) and (3), we get

$$
\begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ \left(c_1 \cos\left(\frac{\lambda}{a}\right) + c_2 \sin\left(\frac{\lambda}{a}\right)\right)T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_1 T(t) = 0 \quad \forall t > 0 \\ c_2 \sin\left(\frac{\lambda}{a}\right)T(t) = 0 \quad \forall t > 0 \end{cases}
$$

If $\sin\left(\frac{\lambda}{a}\right) \neq 0$, then $c_2 T(t) = 0$ $\forall t > 0$ and then $u(x,t) \equiv 0$.

If $\sin\left(\frac{\lambda}{a}\right) = 0$, we obtain $\frac{\lambda}{a} = n\pi$, i.e. $\lambda = n\pi a$, $n = 1, 2, ...$ and

$$u(x,t) = c_2 \sin\left(\frac{\lambda}{a}x\right)(c_3\cos(\lambda t) + c_4\sin(\lambda t)) \tag{16}$$

$$\frac{\partial u}{\partial t}(x,t) = c_2\lambda \sin\left(\frac{\lambda}{a}x\right)(-c_3\sin(\lambda t) + c_4\cos(\lambda t)). \tag{17}$$

Using now the boundary condition (4) and (17), we get

$$0 = \lambda c_2 c_4 \sin\left(\frac{\lambda}{a}x\right), \ \forall 0 < x < \pi.$$

This leads by (16) to

$$u(x,t) = c_2 c_3 \sin\left(\frac{\lambda}{a}x\right)\cos(\lambda t).$$

Therefore all product solutions of (1), (2), (3) and (4) are given in this case by

$$u_n(x,t) = B_n \sin(n\pi x)\cos(n\pi a t), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\cos(n\pi a t) \tag{18}$$

is also a solution of (1), (2), (3) and (4).
Now it is enough to find the coefficients B_n such that the function given in (18) is also a solution of (5) i.e.

$$10^{-2}\sin(3\pi x) = \sum_{n=1}^{\infty} B_n \sin(nx), \quad 0 < x < 1. \tag{19}$$

It is clear that (19) is the half-range expansion of the function $10^{-2}\sin(3\pi x)$ in a sine series. It follows that

$$
\begin{aligned}
B_n &= \frac{2}{1}\int_0^1 10^{-2}\sin(3\pi x)\sin(n\pi x)dx \\
&= 2\int_0^1 10^{-2}\sin(3\pi x)\sin(n\pi x)dx \\
&= 0 \text{ if } n \neq 3 \qquad (20)\\
&= 10^{-2}\int_0^1 2\sin^2(3\pi x)dx \text{ if } n = 3 \\
&= 10^{-2}\int_0^1 (1-\cos(6\pi x)dx \\
&= 10^{-2}\Big[x - \frac{1}{6\pi}\sin(6\pi x)\Big]_0^1 \\
&= 10^{-2}. \qquad (21)
\end{aligned}
$$

Hence we obtain from (18), (20) and (21) the solution of our BVP

$$u(x,t) = 10^{-2}\sin(3\pi x)\cos(3\pi a t).$$

□

8. We would like to solve the following boundary-value problem

$$
\begin{aligned}
& a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{for all} \quad 0 < x < L, t > 0 && (1)\\
& \frac{\partial u}{\partial x}(0,t) = 0,\ t > 0 && (2)\\
& \frac{\partial u}{\partial x}(L,t) = 0,\ t > 0 && (3)\\
& \frac{\partial u}{\partial t}(x,0) = 0,\ 0 < x < L && (4)\\
& u(x,0) = x,\ 0 < x < L. && (5)
\end{aligned}
$$

Let us find all product solutions of the boundary-value problem (1), (2), (3) and (4). Indeed let $u(x,y) = X(x)T(t)$ be such a product solution. Then we have

$$a^2 X''(x)T(t) = X(x)T''(t) \quad \text{for all} \quad x, t. \tag{6}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (6)

$$a^2 \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \textit{constant } k \quad \text{for all} \quad x, t. \tag{7}$$

We deduce from (7) that

$$X''(x) = \frac{k}{a^2} X(x) \tag{8}$$

$$T''(t) = kT(t). \tag{9}$$

The solutions of (8) and (9) depend on the sign of k, i.e. we have

$$X(x) = c_1 x + c_2 \text{ if } k = 0 \tag{10}$$

$$X(x) = c_1 \cosh\left(\frac{\sqrt{k}}{a} x\right) + c_2 \sinh\left(\frac{\sqrt{k}}{a} x\right) \text{ if } k > 0 \tag{11}$$

$$X(x) = c_1 \cos\left(\frac{\sqrt{-k}}{a} x\right) + c_2 \sin\left(\frac{\sqrt{-k}}{a} x\right) \text{ if } k < 0. \tag{12}$$

$$T(t) = c_3 t + c_4 \text{ if } k = 0 \tag{13}$$

$$T(t) = c_3 \cosh(\sqrt{k} t) + c_4 \sinh(\sqrt{k} t) \text{ if } k > 0 \tag{14}$$

$$T(t) = c_3 \cos(\sqrt{-k} t) + c_4 \sin(\sqrt{-k} t) \text{ if } k < 0. \tag{15}$$

We discuss three cases:

Case 1: $k = 0$

In this case we have by (10) and (13) $u(x,t) = X(x)T(t) = (c_1 x + c_2)(c_3 t + c_4)$, which leads to $\frac{\partial u}{\partial x}(x,t) = c_1 T(t)$ and $\frac{\partial u}{\partial t}(x,t) = c_3 X(x)$ Using (2), (3) and (4), we get

$$\begin{cases} c_1(c_3 t + c_4) = 0 & \forall t > 0 \\ c_3(c_1 x + c_2) = 0 & \forall 0 < x < L \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 c_3 = c_1 c_4 = 0 \\ c_1 c_3 = c_2 c_3 = 0 \end{cases} \quad \Rightarrow u(x,t) = c_2 c_4.$$

Hence $u_0(x,t) = A_0$ is a product solution to (1), (2), (3) and (4).

Case 2: $k = \lambda^2 > 0$

In this case we have by (11) and (14) that

$$\begin{aligned}
u(x,t) = X(x)T(t) &= \Big(c_1 \cosh\Big(\frac{\lambda}{a}x\Big) + c_2 \sinh\Big(\frac{\lambda}{a}x\Big)\Big)(c_3 \cosh(\lambda t) + c_4 \sinh(\lambda t))\\
\frac{\partial u}{\partial x}(x,t) &= \frac{\lambda}{a}\Big(c_1 \sinh\Big(\frac{\lambda}{a}x\Big) + c_2 \cosh\Big(\frac{\lambda}{a}x\Big)\Big)(c_3 \cosh(\lambda t) + c_4 \sinh(\lambda t))\\
\frac{\partial u}{\partial t}(x,t) &= \lambda\Big(c_1 \cosh\Big(\frac{\lambda}{a}x\Big) + c_2 \sinh\Big(\frac{\lambda}{a}x\Big)\Big)(c_3 \sinh(\lambda t) + c_4 \cosh(\lambda t)).
\end{aligned}$$

Using (2) and (3), we get

$$\begin{cases} c_2 T(t) = 0 \quad \forall t > 0 \\ \Big(c_1 \sinh\Big(\frac{\lambda}{a}L\Big) + c_2 \cosh\Big(\frac{\lambda}{a}L\Big)\Big)T(t) = 0 \quad \forall t > 0 \end{cases} \Rightarrow \begin{cases} c_2 T(t) = 0 \quad \forall t > 0 \\ c_1 T(t) = 0 \quad \forall t > 0. \end{cases}$$

Hence $u(x,t) \equiv 0$.

<u>*Case 3*: $k = -\lambda^2 < 0$</u>

In this case we have by (12) and (15) that

$$\begin{aligned}
u(x,t) = X(x)T(t) &= \Big(c_1 \cos\Big(\frac{\lambda}{a}x\Big) + c_2 \sin\Big(\frac{\lambda}{a}x\Big)\Big)(c_3 \cos(\lambda t) + c_4 \sin(\lambda t))\\
\frac{\partial u}{\partial x}(x,t) &= \frac{\lambda}{a}\Big(- c_1 \sin\Big(\frac{\lambda}{a}x\Big) + c_2 \cos\Big(\frac{\lambda}{a}x\Big)\Big)(c_3 \cos(\lambda t) + c_4 \sin(\lambda t))\\
\frac{\partial u}{\partial t}(x,t) &= \lambda\Big(c_1 \cos\Big(\frac{\lambda}{a}x\Big) + c_2 \sin\Big(\frac{\lambda}{a}x\Big)\Big)(-c_3 \sin(\lambda t) + c_4 \cos(\lambda t)).
\end{aligned}$$

Using (2) and (3), we get

$$\begin{cases} \frac{\lambda}{a} c_2 T(t) = 0 \quad \forall t > 0 \\ \frac{\lambda}{a}\Big(- c_1 \sin\Big(\frac{\lambda}{a}L\Big) + c_2 \cos\Big(\frac{\lambda}{a}L\Big)\Big)T(t) = 0 \quad \forall t > 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_2 T(t) = 0 \quad \forall t > 0 \\ c_1 \sin\Big(\frac{\lambda}{a}L\Big)T(t) = 0 \quad \forall t > 0 \end{cases}$$

$$\Rightarrow \begin{cases} u(x,t) = c_1 \cos\Big(\frac{\lambda}{a}x\Big)T(t) \\ c_1 \sin\Big(\frac{\lambda}{a}L\Big)T(t) = 0 \quad \forall t > 0 \end{cases}$$

If $\sin\Big(\frac{\lambda}{a}L\Big) \neq 0$, then $c_1 T(t) = 0$ $\forall t > 0$ and then $u(x,t) \equiv 0$.

If $\sin\left(\frac{\lambda}{a}L\right) = 0$, we obtain $\frac{\lambda}{a}L = n\pi$, i.e. $\lambda = \frac{n\pi a}{L}$, $n = 1, 2, ...$ and

$$u(x,t) = c_1 \cos\left(\frac{\lambda}{a}x\right)(c_3\cos(\lambda t) + c_4\sin(\lambda t))$$
$$\frac{\partial u}{\partial t}(x,t) = \lambda c_1 \cos\left(\frac{\lambda}{a}x\right)(-c_3\sin(\lambda t) + c_4\cos(\lambda t)).$$

Using now the boundary condition (4), we get

$$\lambda c_1 c_4 \cos\left(\frac{\lambda}{a}x\right) = 0, \ \forall 0 < x < L.$$

This leads to

$$u(x,t) = c_1 c_3 \cos\left(\frac{\lambda}{a}x\right)\cos(\lambda t).$$

Therefore all product solutions of (1), (2), (3) and (4) are given in this case by

$$u_n(x,t) = A_n \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right) \tag{16}$$

is also a solution of (1), (2), (3) and (4).
Now it is enough to find the coefficients A_n such that the function given in (16) is also a solution of (5) i.e.

$$x = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right), \quad 0 < x < L. \tag{17}$$

It is clear that (17) is the half-range expansion of the function x in a cosine series. it follows that

$$A_0 = \frac{1}{L}\int_0^L x dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2} \tag{18}$$
$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi}{L}x\right)dx. \tag{19}$$

Integrating by parts, we get

$$\begin{aligned}
\int_0^L x\cos\Big(\frac{n\pi}{L}x\Big)dx &= \Big[x\frac{L}{n\pi}\sin\Big(\frac{n\pi}{L}x\Big)\Big]_0^L - \int_0^L \frac{L}{n\pi}\sin\Big(\frac{n\pi}{L}x\Big)dx \\
&= -\frac{L}{n\pi}\int_0^L \sin\Big(\frac{n\pi}{L}x\Big)dx \\
&= -\frac{L}{n\pi}\Big[-\frac{L}{n\pi}\cos\Big(\frac{n\pi}{L}x\Big)\Big]_0^L \\
&= \frac{L^2}{n^2\pi^2}(\cos(n\pi)-1) \\
&= \frac{L^2}{n^2\pi^2}((-1)^n-1). \qquad (20)
\end{aligned}$$

Taking into account (19) and (20), we deduce that

$$A_n = \frac{2L}{n^2\pi^2}((-1)^n-1). \qquad (21)$$

Hence we obtain from (16), (18) and (21) the solution of our BVP

$$\begin{aligned}
u(x,t) &= \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}((-1)^n-1)\cos\Big(\frac{n\pi}{L}x\Big)\cos\Big(\frac{n\pi a}{L}t\Big) \\
&= \frac{L}{2} - \sum_{n=0}^{\infty}\frac{4L}{(2n+1)^2\pi^2}\cos\Big(\frac{(2n+1)\pi}{L}x\Big)\cos\Big(\frac{(2n+1)\pi a}{L}x\Big).
\end{aligned}$$

□