King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Dr. A. Lyaghfouri
MATH 301/Term 062/Hw#22(13.3)/

2. We would like to solve the heat equation

$$k\frac{\partial^2 u}{\partial x} = \frac{\partial u}{\partial t}$$

subject to the conditions $u(0,t) = u(L,t) = 0, t > 0$ and $u(x,0) = x(L-x), 0 < x < L$. We obtain the boundary-value problem

$$k\frac{\partial^2 u}{\partial x} = \frac{\partial u}{\partial t} \quad \text{for all} \quad 0 < x < L, t > 0 \tag{1}$$

$$u(0,t) = 0, \ \ t > 0 \tag{2}$$

$$u(L,t) = 0, \ \ t > 0 \tag{3}$$

$$u(x,0) = x(L-x), 0 < x < L. \tag{4}$$

We start by finding all nontrivial product solutions of (1) subject to conditions (2) and (3). Indeed let $u(x,y) = X(x)T(t)$ be a product solution of (1). Then we have

$$kX''(x)T(t) = X(x)T'(t) \quad \text{for all} \quad 0 < x < L, t > 0. \tag{5}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (3)

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = constant\ c \quad \text{for all} \quad x, t. \tag{6}$$

We deduce from (6) that

$$X''(x) = cX(x) \tag{7}$$

$$T'(t) = ckT(t). \tag{8}$$

The solution of (7) depends on the sign of c, i.e. we have

$$X(x) = c_1 x + c_2 \text{ if } c = 0 \tag{9}$$

$$X(x) = c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x) \text{ if } c = \lambda^2 > 0 \tag{10}$$

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \text{ if } c = -\lambda^2 < 0. \tag{11}$$

The solution of (8) is given by

$$T(t) = c_3 \text{ if } c = 0 \tag{12}$$
$$T(t) = c_3 e^{k\lambda^2 t} \text{ if } c = \lambda^2 > 0 \tag{13}$$
$$T(t) = c_3 e^{-k\lambda^2 t} \text{ if } c = -\lambda^2 < 0. \tag{14}$$

We discuss three cases:

Case 1: $c = 0$

In this case we have by (9) and (12) $u(x,y) = X(x)T(t) = c_1c_3x + c_2c_3$. Using (2) and (3), we get respectively $c_2c_3 = 0$ and $c_1c_3L + c_2c_3 = 0$, which lead to $c_1c_3 = 0$ and $c_2c_3 = 0$. Hence $u(x,y) \equiv 0$.

Case 2: $c = \lambda^2 > 0$

In this case we have by (10) and (13)

$$u(x,y) = X(x)T(t) = c_3 e^{k\lambda^2 t}\big(c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x)\big). \tag{15}$$

Using (2), we get

$$c_1c_3e^{k\lambda^2 t} = 0, \quad t > 0,$$

which leads by substitution into (15) to

$$u(x,y) = c_2c_3e^{k\lambda^2 t}\sinh(\lambda x). \tag{16}$$

Using (3), we get

$$c_2c_3e^{k\lambda^2 t}\sinh(\lambda L) = 0, \quad t > 0,$$

which leads to $c_2c_3 = 0$, since $e^{k\lambda^2 t}\sinh(\lambda L) > 0$. Hence we get by substituting into (16) $u(x,y) \equiv 0$.

Case 3: $c = -\lambda^2 < 0$

In this case we have by (11) and (14)

$$u(x,y) = X(x)T(t) = c_3 e^{-k\lambda^2 t}\big(c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\big). \tag{17}$$

Using (2), we get

$$c_1c_3e^{-k\lambda^2 t} = 0, \quad t > 0,$$

which leads by substitution into (17) to

$$u(x,y) = c_2c_3e^{-k\lambda^2 t}\sin(\lambda x). \tag{18}$$

Using (3), we get

$$c_2c_3e^{-k\lambda^2 t}\sin(\lambda L) = 0, \quad t > 0,$$

which leads to $c_2c_3\sin(\lambda L) = 0$.
If $c_2c_3 = 0$, then we get from (18) $u(x, y) \equiv 0$.
If $c_2c_3 \neq 0$, then we have $\sin(\lambda L) = 0$. We deduce that $\lambda L = n\pi$, $n = 1, 2, ...$ Hence $\lambda = \frac{n\pi}{L}$, $n = 1, 2, ...$
Therefore all product solutions of (1), (2) and (3) are given by

$$u_n(x, t) = B_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}x\right) \tag{19}$$

is also a solution of (1), (2) and (3).
Now it is enough to find the coefficients B_n such that the function given in (19) is also a solution of (4) i.e.

$$x(L - x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right), \quad 0 < x < L. \tag{20}$$

It is clear that (20) is the half-range expansion of the function $x(L-x)$ in a sine series. it follows that

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L x(L - x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= 2\int_0^L x\sin\left(\frac{n\pi}{L}x\right)dx - \frac{2}{L}\int_0^L x^2\sin\left(\frac{n\pi}{L}x\right)dx. \end{aligned} \tag{21}$$

Integrating by parts, we get

$$\begin{aligned}
\int_0^L x\sin\Big(\frac{n\pi}{L}x\Big)dx &= \Big[-x\frac{L}{n\pi}\cos\Big(\frac{n\pi}{L}x\Big)\Big]_0^L - \int_0^L -\frac{L}{n\pi}\cos\Big(\frac{n\pi}{L}x\Big)dx \\
&= -\frac{L^2}{n\pi}\cos(n\pi) + 0 + \frac{L}{n\pi}\int_0^L \cos\Big(\frac{n\pi}{L}x\Big)dx \\
&= \frac{(-1)^{n+1}L^2}{n\pi} + \frac{L}{n\pi}\int_0^L \cos\Big(\frac{n\pi}{L}x\Big)dx \\
&= \frac{(-1)^{n+1}L^2}{n\pi} + \frac{L}{n\pi}\Big[\frac{L}{n\pi}\sin\Big(\frac{n\pi}{L}x\Big)\Big]_0^L \\
&= \frac{(-1)^{n+1}L^2}{n\pi} + \frac{L^2}{n^2\pi^2}(\sin(n\pi) - 0) \\
&= \frac{(-1)^{n+1}L^2}{n\pi}. \qquad (22)
\end{aligned}$$

Integrating by parts twice, we get

$$\begin{aligned}
\int_0^L x^2\sin\Big(\frac{n\pi}{L}x\Big)dx &= \Big[-x^2\frac{L}{n\pi}\cos\Big(\frac{n\pi}{L}x\Big)\Big]_0^L - \int_0^L -2x\frac{L}{n\pi}\cos\Big(\frac{n\pi}{L}x\Big)dx \\
&= -\frac{L^3}{n\pi}\cos(n\pi) + \frac{2L}{n\pi}\int_0^L x\cos\Big(\frac{n\pi}{L}x\Big)dx \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + \frac{2L}{n\pi}\Big(\Big[x\frac{L}{n\pi}\sin\Big(\frac{n\pi}{L}x\Big)\Big]_0^L - \int_0^L \frac{L}{n\pi}\sin\Big(\frac{n\pi}{L}x\Big)dx\Big) \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^2}{n^2\pi^2}(L\sin(n\pi) - 0) - 2\frac{L^2}{n^2\pi^2}\int_0^L \sin\Big(\frac{n\pi}{L}x\Big)dx \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^3}{n^3\pi^3}\Big[\cos\Big(\frac{n\pi}{L}x\Big)\Big]_0^L \\
&= \frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^3}{n^3\pi^3}((-1)^n - 1). \qquad (23)
\end{aligned}$$

Taking into account (21), (22) and (23), we deduce that

$$\begin{aligned}
B_n &= 2\frac{(-1)^{n+1}L^2}{n\pi} - \frac{2}{L}\Big(\frac{(-1)^{n+1}L^3}{n\pi} + 2\frac{L^3}{n^3\pi^3}((-1)^n - 1)\Big) \\
&= \frac{4L^2}{n^3\pi^3}(1 - (-1)^n). \qquad (24)
\end{aligned}$$

Hence we obtain from (19) and (23) the solution of our BVP

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{4L^2(1-(-1)^n)}{n^3\pi^3} e^{-k\frac{n^2\pi^2}{L^2}t} \sin\Big(\frac{n\pi}{L}x\Big) \\ &= \sum_{n=0}^{\infty} \frac{8L^2}{(2n+1)^3\pi^3} e^{-k\frac{(2n+1)^2\pi^2}{L^2}t} \sin\Big(\frac{(2n+1)\pi}{L}x\Big). \end{aligned}$$

□

3. We would like to find the temperature $u(x,t)$ in a rod of length L if the initial temperature is $u(x,0) = f(x)$ $0 < x < L$ and if the ends $x = 0$ and $x = L$ are insulated i.e. $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0,\ t > 0$. This is equivalent to solve the following boundary-value problem

$$k\frac{\partial^2 u}{\partial x} = \frac{\partial u}{\partial t} \text{ for all } 0 < x < L,\ t > 0 \tag{1}$$
$$\frac{\partial u}{\partial x}(0,t) = 0,\ t > 0 \tag{2}$$
$$\frac{\partial u}{\partial x}(L,t) = 0,\ t > 0 \tag{3}$$
$$u(x,0) = f(x),\ 0 < x < L. \tag{4}$$

We start by finding all nontrivial product solutions of (1) subject to conditions (2) and (3). Indeed let $u(x,y) = X(x)T(t)$ be a product solution of (1). Then we have

$$kX''(x)T(t) = X(x)T'(t) \text{ for all } 0 < x < L, t > 0. \tag{5}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (3)

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = constant\ c \text{ for all } x, t. \tag{6}$$

We deduce from (6) that

$$X''(x) = cX(x) \tag{7}$$
$$T'(t) = ckT(t). \tag{8}$$

The solution of (7) depends on the sign of c, i.e. we have

$$\begin{aligned} X(x) &= c_1 x + c_2 \text{ if } c = 0 & (9)\\ X(x) &= c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x) \text{ if } c = \lambda^2 > 0 & (10)\\ X(x) &= c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \text{ if } c = -\lambda^2 < 0. & (11) \end{aligned}$$

The solution of (8) is given by

$$\begin{aligned} T(t) &= c_3 \text{ if } c = 0 & (12)\\ T(t) &= c_3 e^{k\lambda^2 t} \text{ if } c = \lambda^2 > 0 & (13)\\ T(t) &= c_3 e^{-k\lambda^2 t} \text{ if } c = -\lambda^2 < 0. & (14) \end{aligned}$$

We discuss three cases:

Case 1: $c = 0$

In this case we have by (9) and (12) $u(x, y) = X(x)T(t) = c_1c_3x + c_2c_3$. Using (2) and (3), we get -since $\frac{\partial u}{\partial x}(x, t) = c_1c_3$- $c_1c_3 = 0$, which leads to $u(x, y) = c_2c_3$. Hence $u_0(x, y) = A_0$ is a product solution of (1), (2) and (3).

Case 2: $c = \lambda^2 > 0$

In this case we have by (10) and (13)

$$u(x, y) = X(x)T(t) = c_3 e^{k\lambda^2 t}\big(c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x)\big). \tag{15}$$

This leads to

$$\frac{\partial u}{\partial x}(x, y) = \lambda c_3 e^{k\lambda^2 t}\big(c_1 \sinh(\lambda x) + c_2 \cosh(\lambda x)\big). \tag{16}$$

Using (2) and (16), we get

$$\lambda c_2 c_3 e^{k\lambda^2 t} = 0, \quad t > 0,$$

Substituting into (15), we get since $\lambda > 0$

$$u(x, y) = c_1 c_3 e^{k\lambda^2 t} \cosh(\lambda x) \text{ and } \frac{\partial u}{\partial x}(x, y) = \lambda c_1 c_3 e^{k\lambda^2 t} \sinh(\lambda x). \tag{17}$$

Using (3) and (17), we get

$$\lambda c_1 c_3 e^{k\lambda^2 t} \cosh(\lambda L) = 0, \quad t > 0,$$

which leads to $c_1c_3 = 0$, since $\lambda e^{k\lambda^2 t} \cosh(\lambda L) > 0$. Hence we get by substituting into (17) $u(x, y) \equiv 0$.

Case 3: $c = -\lambda^2 < 0$

In this case we have by (11) and (14)

$$u(x,y) = X(x)T(t) = c_3 e^{-k\lambda^2 t}\big(c_1\cos(\lambda x) + c_2\sin(\lambda x)\big). \tag{18}$$

This leads to

$$\frac{\partial u}{\partial x}(x,y) = \lambda c_3 e^{k\lambda^2 t}\big(-c_1\sin(\lambda x) + c_2\cos(\lambda x)\big). \tag{19}$$

Using (2) and (19), we get

$$\lambda c_2 c_3 e^{-k\lambda^2 t} = 0, \quad t > 0,$$

Substituting into (18), we get since $\lambda > 0$

$$u(x,y) = c_1 c_3 e^{-k\lambda^2 t}\cos(\lambda x) \ \text{ and } \ \frac{\partial u}{\partial x}(x,y) = -\lambda c_1 c_3 e^{k\lambda^2 t}\sin(\lambda x). \tag{20}$$

Using (3) and (20), we get

$$-\lambda c_1 c_3 e^{k\lambda^2 t}\sin(\lambda x) = 0, \quad t > 0,$$

which leads to $c_1 c_3 \sin(\lambda L) = 0$.

If $c_1 c_3 = 0$, then we get from (20) $u(x,y) \equiv 0$.

If $c_1 c_3 \neq 0$, then we have $\sin(\lambda L) = 0$. We deduce that $\lambda L = n\pi$, $n = 1, 2, ...$ Hence $\lambda = \frac{n\pi}{L}$, $n = 1, 2, ...$

Therefore all product solutions of (1), (2) and (3) are given in this case by

$$u_n(x,t) = A_n e^{-k\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi}{L}x\right) \tag{21}$$

is also a solution of (1), (2) and (3).

Now it is enough to find the coefficients A_n such that the function given in (21) is also a solution of (4) i.e.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right), \quad 0 < x < L. \tag{22}$$

It is clear that (22) is the half-range expansion of the function $f(x)$ in a cosine series. it follows that

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx. \end{aligned}$$

Hence we get

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\right)e^{-k\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi}{L}x\right).$$

□

6. We consider the following boundary-value problem

$$k\frac{\partial^2 u}{\partial x} - hu = \frac{\partial u}{\partial t} \quad \text{for all} \quad 0 < x < L, \ t > 0 \tag{1}$$

$$u(0,t) = 0, \ t > 0 \tag{2}$$

$$u(L,t) = 0, \ t > 0 \tag{3}$$

$$u(x,0) = f(x), \ 0 < x < L. \tag{4}$$

We start by finding all nontrivial product solutions of (1) subject to conditions (2) and (3). Indeed let $u(x,y) = X(x)T(t)$ be a product solution of (1). Then we have

$$kX''(x)T(t) - hX(x)T(t) = X(x)T'(t) \quad \text{for all} \quad 0 < x < L, t > 0. \tag{5}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (3)

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} + \frac{h}{k} = constant \ c \quad \text{for all} \quad x, t. \tag{6}$$

We deduce from (6) that

$$X''(x) = cX(x) \tag{7}$$

$$T'(t) = (kc - h)T(t). \tag{8}$$

The solution of (7) depends on the sign of c, i.e. we have

$$\begin{aligned} X(x) &= c_1 x + c_2 \text{ if } c = 0 && (9)\\ X(x) &= c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x) \text{ if } c = \lambda^2 > 0 && (10)\\ X(x) &= c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \text{ if } c = -\lambda^2 < 0. && (11)\end{aligned}$$

The solution of (8) is given by

$$\begin{aligned} T(t) &= c_3 e^{-ht} \text{ if } c = 0 && (12)\\ T(t) &= c_3 e^{(k\lambda^2 - h)t} \text{ if } c = \lambda^2 > 0 && (13)\\ T(t) &= c_3 e^{-(k\lambda^2 + h)t} \text{ if } c = -\lambda^2 < 0. && (14)\end{aligned}$$

We discuss three cases:

Case 1: $c = 0$

In this case we have by (9) and (12) $u(x,y) = X(x)T(t) = c_1 c_3 e^{-ht} x + c_2 c_3 e^{-ht}$. Using (2) and (3), we get respectively $c_2 c_3 e^{-ht} = 0$ and $c_1 c_3 L e^{-ht} + c_2 c_3 e^{-ht} = 0$, which lead to $c_2 c_3 = 0$ and $c_1 c_3 L + c_2 c_3 = 0$. Then we get $c_1 c_3 = 0$ and $c_2 c_3 = 0$. Hence $u(x,y) \equiv 0$.

Case 2: $c = \lambda^2 > 0$

In this case we have by (10) and (13)

$$u(x,y) = X(x)T(t) = c_3 e^{(k\lambda^2 - h)t}\big(c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x)\big). \quad (15)$$

Using (2), we get

$$c_1 c_3 e^{(k\lambda^2 - h)t} = 0, \quad t > 0,$$

which leads by substitution into (15) to

$$u(x,y) = c_2 c_3 e^{(k\lambda^2 - h)t} \sinh(\lambda x). \quad (16)$$

Using (3), we get

$$c_2 c_3 e^{(k\lambda^2 - h)t} \sinh(\lambda L) = 0, \quad t > 0,$$

which leads to $c_2 c_3 = 0$, since $e^{(k\lambda^2 - h)t} \sinh(\lambda L) > 0$. Hence we get by substituting into (16) $u(x,y) \equiv 0$.

Case 3: $c = -\lambda^2 < 0$

In this case we have by (11) and (14)

$$u(x,y) = X(x)T(t) = c_3 e^{-(k\lambda^2 + h)t}\big(c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\big). \quad (17)$$

Using (2), we get

$$c_1c_3e^{-(k\lambda^2+h)t} = 0, \quad t > 0,$$

which leads by substitution into (17) to

$$u(x,y) = c_2c_3e^{-(k\lambda^2+h)t}\sin(\lambda x). \tag{18}$$

Using (3), we get

$$c_2c_3e^{-(k\lambda^2+h)t}\sin(\lambda L) = 0, \quad t > 0,$$

which leads to $c_2c_3\sin(\lambda L) = 0$.
If $c_2c_3 = 0$, then we get from (18) $u(x,y) \equiv 0$.
If $c_2c_3 \neq 0$, then we have $\sin(\lambda L) = 0$. We deduce that $\lambda L = n\pi$, $n = 1, 2, ...$ Hence $\lambda = \frac{n\pi}{L}$, $n = 1, 2, ...$
Therefore all product solutions of (1), (2) and (3) are given by

$$u_n(x,t) = B_n e^{-(k\frac{n^2\pi^2}{L^2}+h)t}\sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, ...$$

According to the superposition principle, we know that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-(k\frac{n^2\pi^2}{L^2}+h)t}\sin\left(\frac{n\pi}{L}x\right) \tag{19}$$

is also a solution of (1), (2) and (3).
Now it is enough to find the coefficients B_n such that the function given in (19) is also a solution of (4) i.e.

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right), \quad 0 < x < L. \tag{20}$$

It is clear that (20) is the half-range expansion of the function $f(x)$ in a sine series. it follows that

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \tag{21}$$

Hence we obtain from (19) and (21) the solution of our BVP

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right)e^{-(k\frac{n^2\pi^2}{L^2}+h)t}\sin\left(\frac{n\pi}{L}x\right).$$

□