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MATH 301/Term 062/Hw#21(13.1)/

1. We would like to find all product solutions of the following partial differential equation
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}. \tag{1}$$
Indeed let $u(x, y) = X(x)Y(y)$ be a product solution of (1). Then we have
$$X'(x)Y(y) = X(x)Y'(y) \text{ for all } x, y. \tag{2}$$
If $X(x) \neq 0$ and $Y(y) \neq 0$, we get from (2)
$$\frac{X'(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = constant\ k \text{ for all } x, y. \tag{3}$$
We deduce from (3) that
$$X(x) = c_1 e^{kx} \tag{4}$$
$$Y(y) = c_2 e^{ky}. \tag{5}$$
Taking into account (4) and (5), it follows that all product solutions of (1) are given by
$$u(x, y) = Ce^{kx}e^{ky} = Ce^{k(x+y)}, \text{ where } k \text{ and } C \text{ are arbitrary constants.}$$
□

8. We would like to find all product solutions of the following partial differential equation
$$y\frac{\partial^2 u}{\partial x \partial y} + u = 0. \tag{1}$$
Indeed let $u(x, y) = X(x)Y(y)$ be a product solution of (1). Then we have
$$yX'(x)Y'(y) + X(x)Y(y) = 0 \text{ for all } x, y. \tag{2}$$

If $X(x) \neq 0$ and $yY'(y) \neq 0$, we get from (2)

$$\frac{X'(x)}{X(x)} = -\frac{Y(y)}{yY'(y)} = constant\ k \quad \text{for all} \quad x, y. \tag{3}$$

We deduce from (3) that

$$X'(x) = kX(x) \tag{4}$$
$$Y'(y) = -\frac{1}{ky}Y(y). \tag{5}$$

Solving (4) and (5), we get

$$X(x) = c_1 e^{kx} \tag{6}$$
$$Y(y) = c_2 e^{-\int \frac{1}{ky}dy} = c_2 e^{-\frac{1}{k}\ln|y|} = c_2|y|^{-\frac{1}{k}}. \tag{7}$$

Taking into account (6) and (7), it follows that all product solutions of (1) are given by

$$u(x,y) = Ce^{kx}|y|^{-\frac{1}{k}}, \quad \text{where } k \text{ and } C \text{ are arbitrary constants, with } k \neq 0.$$

□

13. We would like to find all product solutions of the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2k\frac{\partial u}{\partial t} = 0, \quad k > 0. \tag{1}$$

Indeed let $u(x,y,t) = X(x)Y(y)T(t)$ be a product solution of (1). Then we have

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) + 2kX(x)Y(y)T'(t) = 0 \quad \text{for all} \quad x, y, t. \tag{2}$$

If $X(x) \neq 0$, $Y(y) \neq 0$ and $T(t) \neq 0$, we get from (2)

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - 2k\frac{T'(t)}{T(t)} = constant\ a \quad \text{for all} \quad x, y, t. \tag{3}$$

We deduce from (3) that

$$X''(x) = aX(x) \tag{4}$$
$$\frac{Y''(y)}{Y(y)} = -2k\frac{T'(t)}{T(t)} - a = constant\ b. \tag{5}$$

Then (5) leads to

$$
\begin{aligned}
Y''(y) &= bY(y) && (6) \\
\frac{T'(t)}{T(t)} &= \frac{a+b}{2k} = c. && (7)
\end{aligned}
$$

The solutions of (4) and (6) depend on the signs of a and b, i.e. we have

$$
\begin{aligned}
X(x) &= c_1 x + c_2 \text{ if } a = 0 && (8) \\
X(x) &= c_1 \cosh(\sqrt{a}x) + c_2 \sinh(\sqrt{a}x) \text{ if } a > 0 && (9) \\
X(x) &= c_1 \cos(\sqrt{-a}x) + c_2 \sin(\sqrt{-a}x) \text{ if } a < 0. && (10)
\end{aligned}
$$

$$
\begin{aligned}
Y(y) &= c_3 y + c_4 \text{ if } a = 0 && (11) \\
Y(y) &= c_3 \cosh(\sqrt{b}y) + c_4 \sinh(\sqrt{b}y) \text{ if } b > 0 && (12) \\
Y(y) &= c_3 \cos(\sqrt{-b}y) + c_4 \sin(\sqrt{-b}y) \text{ if } b < 0. && (13)
\end{aligned}
$$

The solution of (7) is given by

$$T(t) = c_5 e^{\frac{a+b}{2k}t}. \quad (14)$$

c_1, c_2, c_3, c_4 and c_5 are arbitrary constants.
Taking into account (8)-(14), it follows that all product solutions of (1) are given by

$$
\begin{aligned}
u(x,y,t) &= (c_1 x + c_2)(c_3 y + c_4) \\
u(x,y,t) &= (c_1 x + c_2)(c_3 \cosh(\sqrt{b}y) + c_4 \sinh(\sqrt{b}y) e^{\frac{a}{2k}t} \quad \text{with } a = 0 \text{ and } b > 0 \\
u(x,y,t) &= (c_1 x + c_2)(c_3 \cos(\sqrt{-b}y) + c_4 \sin(\sqrt{-b}y)) e^{\frac{a}{2k}t} \quad \text{with } a = 0 \text{ and } b < 0 \\
u(x,y,t) &= (c_1 \cosh(\sqrt{a}x) + c_2 \sinh(\sqrt{a}x))(c_3 y + c_4) e^{\frac{a}{2k}t} \quad \text{with } a > 0 \text{ and } b < 0 \\
u(x,y,t) &= (c_1 \cosh(\sqrt{a}x) + c_2 \sinh(\sqrt{a}x))(c_3 \cos(\sqrt{-b}y) + c_4 \sin(\sqrt{-b}y)) e^{\frac{a+b}{2k}t} \\
&\quad \text{with } a > 0 \text{ and } b < 0 \\
u(x,y,t) &= (c_1 \cosh(\sqrt{a}x) + c_2 \sinh(\sqrt{a}x))(c_3 \cosh(\sqrt{-b}y) + c_4 \sinh(\sqrt{-b}y)) e^{\frac{a+b}{2k}t} \\
&\quad \text{with } a > 0 \text{ and } b > 0 \\
u(x,y,t) &= (c_1 \cos(\sqrt{a}x) + c_2 \sin(\sqrt{a}x))(c_3 y + c_4) e^{\frac{a}{2k}t} \quad \text{with } a < 0 \text{ and } b = 0 \\
u(x,y,t) &= (c_1 \cos(\sqrt{a}x) + c_2 \sin(\sqrt{a}x))(c_3 \cos(\sqrt{-b}y) + c_4 \sin(\sqrt{-b}y)) e^{\frac{a+b}{2k}t} \\
&\quad \text{with } a < 0 \text{ and } b < 0 \\
u(x,y,t) &= (c_1 \cos(\sqrt{a}x) + c_2 \sin(\sqrt{a}x))(c_3 \cosh(\sqrt{-b}y) + c_4 \sinh(\sqrt{-b}y)) e^{\frac{a+b}{2k}t} \\
&\quad \text{with } a < 0 \text{ and } b > 0.
\end{aligned}
$$

□

16. We would like to use the method of separation of variables to find a family of solutions of the following partial differential equation

$$a^2\frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad \text{where } g \text{ is a constant.} \tag{1}$$

First we remark that u satisfies (1) if and only if the function $v(x,y) = u(x,y) - \frac{1}{2a^2}x^2$ satisfies

$$a^2\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}. \tag{2}$$

Let us find all product solutions of the partial differential equation (2)
Indeed let $v(x,y) = X(x)T(t)$ be a product solution of (2). Then we have

$$a^2X''(x)T(t) = X(x)T''(t) \text{ for all } x, t. \tag{3}$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we get from (3)

$$a^2\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \mathit{constant}\ k \text{ for all } x, t. \tag{4}$$

We deduce from (4) that

$$X''(x) = \frac{k}{a^2}X(x) \tag{5}$$

$$T''(t) = kT(t). \tag{6}$$

The solutions of (5) and (6) depend on the sign of k, i.e. we have

$$X(x) = c_1x + c_2 \text{ if } k = 0 \tag{7}$$

$$X(x) = c_1\cosh\left(\frac{\sqrt{k}}{a}x\right) + c_2\sinh\Big)\Big(\frac{\sqrt{k}}{a}x\Big)\Big) \text{ if } k > 0 \tag{8}$$

$$X(x) = c_1\cos\left(\frac{\sqrt{-k}}{a}x\right) + c_2\sin\left(\frac{\sqrt{-k}}{a}x\right) \text{ if } k < 0. \tag{9}$$

$$Y(y) = c_3y + c_4 \text{ if } k = 0 \tag{10}$$

$$Y(y) = c_3\cosh(\sqrt{k}y) + c_4\sinh(\sqrt{k}y) \text{ if } k > 0 \tag{11}$$

$$Y(y) = c_3\cos(\sqrt{-k}y) + c_4\sin(\sqrt{-k}y) \text{ if } k < 0. \tag{12}$$

Therefore all product solutions of (2) are given by

$$v(x,y) = (c_1x + c_2)(c_3y + c_4) \text{ if } k = 0$$
$$v(x,y) = \left(c_1 \cosh\left(\frac{\sqrt{k}}{a}x\right) + c_2 \sinh\right)\left(\frac{\sqrt{k}}{a}x\right)\Big)(c_3 \cosh(\sqrt{k}y) + c_4 \sinh(\sqrt{k}y)) \text{ if } k > 0$$
$$v(x,y) = \left(c_1 \cos\left(\frac{\sqrt{-k}}{a}x\right) + c_2 \sin\left(\frac{\sqrt{-k}}{a}x\right)\right)(c_3 \cos(\sqrt{-k}y) + c_4 \sin(\sqrt{-k}y)) \text{ if } k < 0.$$

Hence we obtain the solutions of (1)

$$u(x,y) = \frac{1}{2a^2}x^2 + (c_1x + c_2)(c_3y + c_4)$$
$$u(x,y) = \frac{1}{2a^2}x^2 + \left(c_1 \cosh\left(\frac{\sqrt{k}}{a}x\right) + c_2 \sinh\right)\left(\frac{\sqrt{k}}{a}x\right)\Big)(c_3 \cosh(\sqrt{k}y) + c_4 \sinh(\sqrt{k}y))$$
$$\text{with } k > 0$$
$$u(x,y) = \frac{1}{2a^2}x^2 + \left(c_1 \cos\left(\frac{\sqrt{-k}}{a}x\right) + c_2 \sin\left(\frac{\sqrt{-k}}{a}x\right)\right)(c_3 \cos(\sqrt{-k}y) + c_4 \sin(\sqrt{-k}y))$$
$$\text{with } k < 0.$$

□

20. We consider the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

This is a partial differential equation of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

where $A = 1,\ B = -1,\ C = -3,\ D = E = F = 0$.
Since $B^2 - 4AC = (-1)^2 - 4(1)(-3) = 1 + 12 = 13 > 0$, the partial differential equation (1) is a hyperbolic equation. □

26. We consider the following partial differential equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0. \tag{1}$$

This is a partial differential equation of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial t} + Fu = 0,$$

where $A = k$, $B = C = D = 0$, $E = -1$, $F = 0$.
Since $B^2 - 4AC = 0^2 - 4k(0) = 0$, the partial differential equation (1) is a parabolic equation. □

28. We consider the following partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{1}$$

We would like to verify that $u(r,\theta) = (c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))(c_3r^\alpha + c_4r^{-\alpha})$ is a solution of (1). Indeed we have

$$\frac{\partial u}{\partial r} = \alpha(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))(c_3r^{\alpha-1} - c_4r^{-\alpha-1}) \tag{2}$$
$$\frac{\partial^2 u}{\partial r^2} = \alpha(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))((\alpha-1)c_3r^{\alpha-2} + (\alpha+1)c_4r^{-\alpha-2}) \tag{3}$$
$$\frac{\partial^2 u}{\partial \theta^2} = -\alpha^2(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))(c_3r^\alpha + c_4r^{-\alpha}). \tag{4}$$

We deduce then from (2),(3) and (4) that

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \alpha(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))((\alpha-1)c_3r^{\alpha-2} + (\alpha+1)c_4r^{-\alpha-2}) \\ &\quad +\alpha(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))(c_3r^{\alpha-2} - c_4r^{-\alpha-2}) \\ &\quad -\alpha^2(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))(c_3r^{\alpha-2} + c_4r^{-\alpha-2}) \\ &= \alpha(c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta))\Big((\alpha-1)c_3r^{\alpha-2} + (\alpha+1)c_4r^{-\alpha-2} \\ &\quad +c_3r^{\alpha-2} - c_4r^{-\alpha-2} - \alpha c_3r^{\alpha-2} - \alpha c_4r^{-\alpha-2}\Big) \\ &= 0.\end{aligned}$$

□