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4. We would like to expand the function $f(x) = 1$, $0 < x < 2$, in a Fourier-Bessel series using Bessel functions of order zero that satisfy $J_0'(2\alpha) = 0$. Here we have $b = 2$, $n = 0$ and $h = 0$. The Fourier-Bessel series is given by

$$c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x), \tag{1}$$

where

$$c_1 = \frac{2}{2^2}\int_0^2 x f(x)dx = \frac{1}{2}\int_0^2 x dx = \frac{1}{2}\Big[\frac{x^2}{2}\Big]_0^2 = \frac{1}{2}\frac{2^2}{2} = 1. \tag{2}$$

$$\begin{aligned}
c_i &= \frac{2}{2^2 J_0^2(2\alpha_i)}\int_0^2 x J_0(\alpha_i x) f(x)dx = \frac{1}{2J_0^2(2\alpha_i)}\int_0^2 x J_0(\alpha_i x)dx \\
&= \frac{1}{2J_0^2(2\alpha_i)}\int_0^{2\alpha_i} \frac{1}{\alpha_i^2} t J_0(t)dt \\
&= \frac{1}{2\alpha_i^2 J_0^2(2\alpha_i)}\int_0^{2\alpha_i} \frac{d}{dt}[tJ_1(t)]dt \\
&= \frac{1}{2\alpha_i^2 J_0^2(2\alpha_i)}[tJ_1(t)]_0^{2\alpha_i} \\
&= \frac{1}{2\alpha_i^2 J_0^2(2\alpha_i)} 2\alpha_i J_1(2\alpha_i) \\
&= \frac{J_1(2\alpha_i)}{\alpha_i J_0^2(2\alpha_i)}.
\end{aligned} \tag{3}$$

Taking into account (1), (2) and (3), it follows that the Fourier-Bessel series of f on the interval $[0, 2]$ is given by

$$1 + \sum_{i=2}^{\infty} \frac{J_1(2\alpha_i)}{\alpha_i J_0^2(2\alpha_i)} J_0(\alpha_i x).$$

□

6. We would like to expand the function $f(x) = 1$, $0 < x < 2$, in a Fourier-Bessel series using Bessel functions of order zero that satisfy $J_0(2\alpha) + \alpha J_0'(2\alpha) = 0$. Here we have $b = 2$, $n = 0$ and $h = 1$. The Fourier-Bessel series is given by

$$\sum_{i=1}^{\infty} c_i J_0(\alpha_i x), \tag{1}$$

where

$$\begin{aligned}
c_i &= \frac{2\alpha_i^2}{(\alpha_i^2 2^2 - 0^2 + 1^2)J_0^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x) f(x) dx = \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x) dx \\
&= \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{1}{\alpha_i^2} t J_0(t) dt \\
&= \frac{2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{d}{dt}[t J_1(t)] dt \\
&= \frac{2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} [t J_1(t)]_0^{2\alpha_i} \\
&= \frac{2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} 2\alpha_i J_1(2\alpha_i) \\
&= \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)}.
\end{aligned} \tag{2}$$

Taking into account (1) and (2), it follows that the Fourier-Bessel series of f on the interval $[0, 2]$ is given by

$$\sum_{i=1}^{\infty} \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} J_0(\alpha_i x).$$

□

8. We would like to expand the function $f(x) = x^2$, $0 < x < 1$, in a Fourier-Bessel series using Bessel functions of order 2 that satisfy $J_2(\alpha) = 0$. Here we have $b = 1$,

$n = 2$ and $h = 0$. The Fourier-Bessel series is given by

$$\sum_{i=1}^{\infty} c_i J_2(\alpha_i x), \tag{1}$$

where

$$\begin{aligned} c_i &= \frac{2}{1^2 J_3^2(\alpha_i)} \int_0^1 x J_2(\alpha_i x) f(x) dx = \frac{2}{J_3^2(\alpha_i)} \int_0^1 x^3 J_2(\alpha_i x) dx \\ &= \frac{2}{J_3^2(\alpha_i)} \int_0^{\alpha_i} \frac{1}{\alpha_i^4} t^3 J_2(t) dt \\ &= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} \int_0^{\alpha_i} \frac{d}{dt}[t^3 J_3(t)] dt \\ &= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} [t^3 J_3(t)]_0^{\alpha_i} \\ &= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} \alpha_i^3 J_3(\alpha_i) \\ &= \frac{2}{\alpha_i J_3(\alpha_i)}. \end{aligned} \tag{2}$$

Taking into account (1) and (2), it follows that the Fourier-Bessel series of f on the interval $[0, 1]$ is given by

$$\sum_{i=1}^{\infty} \frac{2}{\alpha_i J_3(\alpha_i)} J_2(\alpha_i x).$$

□

15. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad -1 \leq x < 0 \\ x, & \text{if} \quad 0 \leq x \leq 1. \end{cases}$$

The Fourier-Legendre expansion of f on the interval $[-1, 1]$ is given by

$$\sum_{n=0}^{\infty} c_n P_n(x) \tag{1}$$

where

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx = \frac{2n+1}{2}\int_{0}^{1} xP_n(x)dx. \tag{2}$$

We would like to find the first five nonzero terms in (1). We recall that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$ and $P_6(x) = \frac{1}{48}(693x^6 - 945x^4 + 315x^2 - 15)$. So we have from (2)

$$\begin{aligned} c_0 &= \frac{1}{2}\int_0^1 xP_0(x)dx = \frac{1}{2}\int_0^1 x.1dx \\ &= \frac{1}{2}\Big[\frac{x^2}{2}\Big]_0^1 \\ &= \frac{1}{4}. \end{aligned} \tag{3}$$

$$\begin{aligned} c_1 &= \frac{3}{2}\int_0^1 xP_1(x)dx = \frac{3}{2}\int_0^1 x.xdx \\ &= \frac{3}{2}\Big[\frac{x^3}{3}\Big]_0^1 \\ &= \frac{1}{2}. \end{aligned} \tag{4}$$

$$\begin{aligned} c_2 &= \frac{5}{2}\int_0^1 xP_2(x)dx = \frac{5}{2}\int_0^1 x.\frac{1}{2}(3x^2-1)dx \\ &= \frac{5}{4}\Big[\frac{3x^4}{4} - \frac{x^2}{2}\Big]_0^1 \\ &= \frac{5}{4}\Big[\frac{3}{4} - \frac{1}{2}\Big] = \frac{5}{16}. \end{aligned} \tag{5}$$

$$\begin{aligned} c_3 &= \frac{7}{2}\int_0^1 xP_3(x)dx = \frac{7}{2}\int_0^1 x.\frac{1}{2}(5x^3-3x)dx \\ &= \frac{7}{4}[x^5 - x^3]_0^1 \\ &= \frac{7}{4}[1-1] = 0. \end{aligned} \tag{6}$$

$$\begin{aligned} c_4 &= \frac{9}{2}\int_0^1 xP_4(x)dx = \frac{9}{2}\int_0^1 x.\frac{1}{8}(35x^4 - 30x^2 + 3)dx \\ &= \frac{9}{16}\Big[\frac{35x^6}{6} - \frac{30x^4}{4} + \frac{3x^2}{2}\Big]_0^1 \\ &= \frac{9}{16}\Big[\frac{35}{6} - \frac{15}{2} + \frac{3}{2}\Big] = \frac{9}{16}.\frac{-1}{6} = -\frac{3}{32}. \end{aligned} \tag{7}$$

$$\begin{aligned} c_5 &= \frac{11}{2}\int_0^1 xP_5(x)dx = \frac{11}{2}\int_0^1 x.\frac{1}{8}(63x^5 - 70x^3 + 15x)dx \\ &= \frac{11}{16}\Big[\frac{63x^7}{7} - \frac{70x^5}{5} + \frac{15x^3}{3}\Big]_0^1 \\ &= \frac{11}{16}[9 - 14 + 5] = 0. \end{aligned} \tag{8}$$

$$\begin{aligned} c_6 &= \frac{13}{2}\int_0^1 xP_6(x)dx = \frac{13}{2}\int_0^1 x.\frac{1}{48}(693x^6 - 945x^4 + 315x^2 - 15)dx \\ &= \frac{13}{96}\Big[\frac{693x^8}{8} - \frac{945x^6}{6} + \frac{315x^4}{4} - 15x\Big]_0^1 \\ &= \frac{13}{96}\Big[\frac{693}{8} - \frac{945}{6} + \frac{315}{4} - 15\Big] = \frac{13}{96}.\frac{-57}{8} = -\frac{741}{768}. \end{aligned} \tag{9}$$

Taking into account (1), (2), (3),... and (9), it follows that the first five nonzero terms in (1) are given by

$$\frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) - \frac{741}{768}P_6(x)$$

□

20. Let f be an odd function on $[-1, 1]$. The Fourier-Legendre expansion of f on the interval $[-1, 1]$ is given by

$$\sum_{n=0}^{\infty} c_n P_n(x) \tag{1}$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx. \tag{2}$$

We know (see 5.3) that $P_n(x)$ is an even or odd function according to whether n is even or odd. It follows that $f(x)P_{2n}(x)$ is an odd function and $f(x)P_{2n+1}(x)$ is an even function. Therefore we get from (2)

$$c_{2n} = 0 \tag{3}$$

and

$$c_{2n+1} = 2\frac{4n+3}{2} \int_{0}^{1} f(x) P_{2n+1}(x) dx = (4n+3) \int_{0}^{1} f(x) P_{2n+1}(x) dx. \tag{4}$$

Taking into account (1), (3) and (4), the Fourier-Legendre expansion of f on the interval $[-1, 1]$ is given by

$$\sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x).$$

□