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MATH 301/Term 062/Hw#2(9.5)/

4. Let $F(x, y, z) = xy\cos(yz)$ be a function. It is clear that F is defined and differentiable everywhere with:

$$\begin{cases} \frac{\partial f}{\partial x} = y\cos(yz) \\ \frac{\partial f}{\partial y} = x\cos(yz) - xyz\sin(yz), \\ \frac{\partial f}{\partial z} = -xy^2\sin(yz). \end{cases}$$

Hence the gradient of F is given by

$$\nabla F(x, y, z) = y\cos(yz)\mathbf{i} + (x\cos(yz) - xyz\sin(yz))\mathbf{j} - xy^2\sin(yz)\mathbf{k}.$$

□

8. Let $F(x, y, z) = \ln(x^2 + y^2 + z^2)$ be a function. F is defined and differentiable except at $(0, 0, 0)$ with:

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} \\ \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2}, \\ \frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2}. \end{cases}$$

Hence we get for all $(x, y, z) \neq (0, 0, 0)$

$$\nabla F(x, y, z) = \frac{2x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{2y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{2z}{x^2 + y^2 + z^2}\mathbf{k}.$$

and in particular

$$\nabla F(-4, 3, 5) = -\frac{8}{50}\mathbf{i} + \frac{6}{50}\mathbf{j} + \frac{10}{50}\mathbf{k} = -\frac{4}{25}\mathbf{i} + \frac{3}{25}\mathbf{j} + \frac{1}{5}\mathbf{k}.$$

□

10. Let $f(x,y) = 3x - y^2$ be a function. We would like to find $D_u f(x,y)$-using Definition 9.5-, where u is the unit vector

$$u = \cos(\pi/4)\mathbf{i} + \sin(\pi/4)\mathbf{j} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}.$$

$$\begin{aligned}
D_u f(x,y) &= \lim_{h\to 0} \frac{f(x + h\frac{\sqrt{2}}{2}, y + h\frac{\sqrt{2}}{2}) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{3(x + h\frac{\sqrt{2}}{2}) - (y + h\frac{\sqrt{2}}{2})^2 - 3x + y^2}{h} \\
&= \lim_{h\to 0} \frac{3h\frac{\sqrt{2}}{2} - hy\sqrt{2} - \frac{h^2}{2}}{h} \\
&= \lim_{h\to 0} 3\frac{\sqrt{2}}{2} - y\sqrt{2} - \frac{h}{2} = 3\frac{\sqrt{2}}{2} - y\sqrt{2}.
\end{aligned}$$

□

14. Let $f(x,y) = \frac{xy}{x+y}$ be a function. We would like to find $D_u f(x,y)$, where u is the vector $u = 6\mathbf{i} + 8\mathbf{j}$. Since u is not a unit vector we have

$$D_u f(x,y) = \nabla f(x,y).\frac{1}{||u||}u.$$

Now

$$\begin{aligned}
&\nabla f(x,y) = \frac{y^2}{(x+y)^2}\mathbf{i} + \frac{x^2}{(x+y)^2}\mathbf{j} \\
&\nabla f(2,-1) = \mathbf{i} + 4\mathbf{j} \\
&\frac{1}{||u||}u = \frac{1}{\sqrt{36+64}}(6\mathbf{i} + 8\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.
\end{aligned}$$

Hence

$$D_u f(2,-1) = (\mathbf{i} + 4\mathbf{j}).(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}) = \frac{19}{5}.$$

□

22. Let $f(x, y) = x^3 - 5xy + y^2$ be a function and let Π be the plane that passes through the points $P(1, 1)$ and $Q(-1, 6)$ and is perpendicular to the $xy-$plane. The slope of the tangent line at the point $(1, 1, -3)$ to the curve C obtained as the intersection of the plane Π and the graph $z = f(x, y)$ of the function f in the direction of $\overrightarrow{QP}$ is given by the directional derivative $D_u f(1, 1)$, where u is the vector $u = \overrightarrow{QP} = 2\mathbf{i} - 5\mathbf{j}$. Since u is not a unit vector we have

$$D_u f(x, y) = \nabla f(x, y).\frac{1}{||u||}u.$$

Now

$$\begin{aligned}
&\nabla f(x, y) = (3x^2 - 5y)\mathbf{i} + (2y - 5x)\mathbf{j} \\
&\nabla f(1, 1) = -2\mathbf{i} - 3\mathbf{j} \\
&\frac{1}{||u||}u = \frac{1}{\sqrt{4 + 25}}(2\mathbf{i} - 5\mathbf{j}) = \frac{2}{\sqrt{29}}\mathbf{i} - \frac{5}{\sqrt{29}}\mathbf{j}.
\end{aligned}$$

Hence the slope

$$D_u f(1, 1) = (-2\mathbf{i} - 3\mathbf{j}).(\frac{2}{\sqrt{29}}\mathbf{i} - \frac{5}{\sqrt{29}}\mathbf{j}) = \frac{11}{\sqrt{29}}.$$

□

30. The function $F(x, y, z) = \ln\left(\frac{xy}{z}\right)$ decreases most rapidly at the point $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ in the direction of the vector $-\nabla F\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$.
Now

$$\begin{aligned}
\nabla F(x, y, z) &= \frac{1}{x}\mathbf{i} + \frac{1}{y}\mathbf{j} - \frac{1}{z}\mathbf{k} \\
\nabla F(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) &= 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}.
\end{aligned}$$

F decreases most rapidly at $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ in the direction of the vector $-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$. The minimum rate is given by

$$-||\nabla F(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})|| = -\sqrt{4 + 36 + 9} = -\sqrt{49} = -7.$$