King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Dr. A. Lyaghfouri

MATH 301/Term 062/Hw#19(12.5)/

2. We would like to find the eigenfunctions and eigenvalues of the following BVP

$$y'' + \lambda y = 0 \tag{1}$$
$$y(0) + y'(0) = 0 \tag{2}$$
$$y(1) = 0. \tag{3}$$

We shall discuss three cases.

Case 1 : $\lambda = 0$

In this case (1) becomes $y'' = 0$ and its general solution is given by $y(x) = c_1 x + c_2$, where c_1 and c_2 are constants. Since $y'(x) = c_1$, we get from (2) and (3) that $c_1 + c_2 = 0$. So $y(x) = c_1(x-1)$. Hence $y(x) = x - 1$ is an eigenfunction corresponding to the eigenvalue 0 of the BVP.

Case 2 : $\lambda = -\alpha^2 < 0, (\alpha = \sqrt{-\lambda})$

In this case (1) becomes $y'' = \alpha^2 y$ and its general solution is given by $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$, where c_1 and c_2 are constants. Since $y'(x) = c_1\alpha \sinh(\alpha x) + c_2 \alpha \cosh(\alpha x)$, we get from 2) and (3) that

$$\begin{cases} c_1 + c_2\alpha = 0 \\ c_1 \cosh(\alpha) + c_2 \sinh(\alpha) = 0. \end{cases} \Leftrightarrow \begin{cases} c_1 = -c_2\alpha \\ -c_2(\alpha\cosh(\alpha) - \sinh(\alpha)) = 0. \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = -c_2\alpha \\ c_2(\alpha - \tanh(\alpha)) = 0. \end{cases} \Leftrightarrow \quad c_2 = c_1 = 0 \quad \text{or} \quad \tanh(\alpha) = \alpha.$$

Given that the equation $\tanh(\alpha) = \alpha$ has 0 as the unique solution, we get $c_1 = c_2 = 0$. Hence $y(x) \equiv 0$ and there is no negative eigenvalue of the BVP.

Case 3 : $\lambda = \alpha^2 > 0, (\alpha = \sqrt{\lambda})$

In this case (1) becomes $y'' = -\alpha^2 y$ and its general solution is given by $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, where c_1 and c_2 are constants. Since $y'(x) = -c_1\alpha \sin(\alpha x) +$

$c_2\alpha\cos(\alpha x)$, we get from 2) and (3) that

$$\begin{cases} c_1 + c_2\alpha = 0 \\ c_1\cos(\alpha) + c_2\sin(\alpha) = 0. \end{cases} \Leftrightarrow \begin{cases} c_1 = -c_2\alpha \\ -c_2(\alpha\cos(\alpha) - \sin(\alpha)) = 0. \end{cases}$$

$$\Leftrightarrow \quad c_2 = c_1 = 0 \quad \text{or} \quad \tan(\alpha) = \alpha.$$

If $c_1 = 0$, then $c_2 = c_1 = 0$ and $y(x) \equiv 0$.

If $c_1 \neq 0$, then $\tan(\alpha) = \alpha$. This equation has an infinite number of positive roots, namely, the $x-$coordinates of the points where the graph of $y = \tan(x)$ intersects with the line $y = x$. Let α_n, $n = 1, 2, ...$ be the positive roots of this equation. Then the eigenvalues of our BVP are given by $\lambda_n = \alpha_n^2$, $n = 1, 2, ...$ The corresponding eigenfunctions are $y_n(x) = \alpha_n\cos(\alpha_n x) - \sin(\alpha_n x)$, $n = 1, 2, ...$

Finally the eigenvalues of our BVP are given by $\lambda_0 = 0$ and $\lambda_n = \alpha_n^2$, $n = 1, 2, ...$ The corresponding eigenfunctions are $y_0(x) = x - 1$ and $y_n(x) = \alpha_n\cos(\alpha_n x) - \sin(\alpha_n x)$, $n = 1, 2, ...$

□

4. We would like to find the eigenfunctions and eigenvalues of the following BVP

$$y'' + \lambda y = 0 \tag{1}$$
$$y(-L) = y(L) \tag{2}$$
$$y'(-L) = y'(L). \tag{3}$$

We shall discuss three cases.

<u>*Case 1* : $\lambda = 0$</u>

In this case (1) becomes $y'' = 0$ and its general solution is given by $y(x) = c_1x + c_2$, where c_1 and c_2 are constants. Since $y'(x) = c_1$, we get from (2) and (3) that $c_1L + c_2 = -c_1L + c_2 \Leftrightarrow 2c_1L = 0 \Leftrightarrow c_1 = 0$. So $y(x) = c_2$. Hence $y(x) = 1$ is an eigenfunction corresponding to the eigenvalue 0 of the BVP.

<u>*Case 2* : $\lambda = -\alpha^2 < 0, (\alpha = \sqrt{-\lambda})$</u>

In this case (1) becomes $y'' = \alpha^2 y$ and its general solution is given by $y(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$, where c_1 and c_2 are constants. Since $y'(x) = c_1\alpha\sinh(\alpha x) + c_2\alpha\cosh(\alpha x)$, we get from 2) and (3) that

$$\begin{cases} c_1\cosh(-\alpha L) + c_2\sinh(-\alpha L) = c_1\cosh(\alpha L) + c_2\sinh(\alpha L) \\ c_1\alpha\sinh(-\alpha L) + c_2\alpha\cosh(-\alpha L) = c_1\alpha\sinh(\alpha L) + c_2\alpha\cosh(\alpha L). \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} c_1 \cosh(\alpha L) - c_2 \sinh(\alpha L) = c_1 \cosh(\alpha L) + c_2 \sinh(\alpha L) \\ -c_1 \alpha \sinh(\alpha L) + c_2 \alpha \cosh(\alpha L) = c_1 \alpha \sinh(\alpha L) + c_2 \alpha \cosh(\alpha L). \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} c_2 \sinh(\alpha L) = 0 \\ c_1 \alpha \sinh(\alpha L) = 0. \end{cases} \quad \Leftrightarrow \quad c_2 = c_1 = 0.$$

Hence $y(x) \equiv 0$ and there is no negative eigenvalue of the BVP.

<u>*Case 3* : $\lambda = \alpha^2 > 0, (\alpha = \sqrt{\lambda})$</u>

In this case (1) becomes $y'' = -\alpha^2 y$ and its general solution is given by $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, where c_1 and c_2 are constants. Since $y'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x)$, we get from 2) and (3) that

$$\begin{cases} c_1 \cos(-\alpha L) + c_2 \sin(-\alpha L) = c_1 \cos(\alpha L) + c_2 \sin(\alpha L) \\ -c_1 \alpha \sin(-\alpha L) + c_2 \alpha \cos(-\alpha L) = -c_1 \alpha \sin(\alpha L) + c_2 \alpha \cos(\alpha L). \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} c_1 \cos(\alpha L) - c_2 \sin(\alpha L) = c_1 \cos(\alpha L) + c_2 \sin(\alpha L) \\ c_1 \alpha \sin(\alpha L) + c_2 \alpha \cos(\alpha L) = -c_1 \alpha \sin(\alpha L) + c_2 \alpha \cos(\alpha L). \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} c_2 \sin(\alpha L) = 0 \\ c_1 \alpha \sin(\alpha L) = 0. \end{cases} \quad \Leftrightarrow \quad c_2 = c_1 = 0 \ \text{ or } \ \sin(\alpha L) = 0.$$

If $c_2 = c_1 = 0$, then $y(x) \equiv 0$.

If $c_1 \neq 0$ or $c_2 \neq 0$, then $\sin(\alpha L) = 0$. This equation has an infinite number of positive roots, namely, $\alpha_n = \frac{n\pi}{L}$, $n = 1, 2, ...$ Then the eigenvalues of our BVP are given by $\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, ...$ Each eigenvalue determines two eigenfunctions $y_n(x) = \cos\left(\frac{n\pi}{L} x\right)$ and $z_n(x) = \sin\left(\frac{n\pi}{L} x\right)$, $n = 1, 2, ...$

Finally the eigenvalues of our BVP are given by $\lambda_0 = 0$ and $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, ...$ The corresponding eigenfunctions are $y_0(x) = 1$, $y_n(x) = \cos\left(\frac{n\pi}{L} x\right)$ and $z_n(x) = \sin\left(\frac{n\pi}{L} x\right)$, $n = 1, 2, ...$

□

6. We consider the BVP

$$\begin{aligned} &y'' + \lambda y = 0 \\ &y(0) = 0 \\ &y(1) + y'(1) = 0. \end{aligned}$$

It has been shown (see Example 2) that the eigenvalues of this BVP are given by $\lambda_n = \alpha_n^2$, $n = 1, 2, ...$, where α_n are the positive roots of the equation $\tan(\alpha) = -\alpha$. The corresponding eigenfunctions are $y_n(x) = \sin(\alpha_n x)$, $n = 1, 2, ...$

Here we want to prove that $||y_n||^2 = \frac{1}{2}(1 + \cos^2(\alpha_n))$.
Indeed

$$\begin{aligned}
||y_n||^2 = &= \int_0^1 \sin^2(\alpha_n x)dx = \frac{1}{2}\int_0^1 (1 - \cos(2\alpha_n x))dx \\
&= \frac{1}{2}\Big[x - \frac{1}{2\alpha_n}\sin(2\alpha_n x)\Big]_0^1 \\
&= \frac{1}{2}\Big(1 - \frac{1}{2\alpha_n}\sin(2\alpha_n)\Big) \\
&= \frac{1}{2}\Big(1 - \frac{1}{\alpha_n}\sin(\alpha_n)\cos(\alpha_n)\Big) \\
&= \frac{1}{2}\Big(1 - \frac{1}{\alpha_n}\tan(\alpha_n)\cos^2(\alpha_n)\Big) \\
&= \frac{1}{2}(1 + \cos^2(\alpha_n)).
\end{aligned}$$

□

12. We consider the parametric Bessel differential equation

$$x^2 y'' + xy + (\lambda x^2 - 1)y = 0. \tag{1}$$

subject to the boundary conditions

$$y \text{ is bounded at } 0, \quad y(3) = 0. \tag{2}$$

a) When $\lambda = \alpha^2$, we know that the general solution of (1) is given by $y(x) = c_1 J_1(\alpha x) + c_2 Y_1(\alpha x)$. Since y is bounded at 0, we must have $c_2 = 0$. Moreover $y(3) = 0$, leads to $J_1(3\alpha) = 0$. Hence the eigenvalues of the BVP (1)-(2) are given by $\lambda_n = \alpha_n^2$, where α_n, $n = 1, 2, ...$ are the roots of the equation $J_1(3\alpha) = 0$. The corresponding eigenfunctions are $y_n(x) = J_1(\alpha_n x)$, $n = 1, 2, ...$