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2. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \le x \le 1 \\ 1, & \text{if} \quad 1 < x \le 2. \end{cases}$$

We would like to expand f in a Fourier series on the interval $[0, 2]$.
The Fourier series expansion of f on the interval $[0, 2]$ is obtained from the Fourier series of the function f^* defined on the interval $[-2, 2]$ by

$$f^*(x) = \begin{cases} f(2+x), & \text{if} \quad -2 \le x < 0 \\ f(x), & \text{if} \quad 0 \le x \le 2. \end{cases}$$

i.e.

$$f^*(x) = \begin{cases} 0, & \text{if} \quad -2 \le x \le -1 \\ 1, & \text{if} \quad -1 < x \le 0. \\ 0, & \text{if} \quad 0 \le x \le 1 \\ 1, & \text{if} \quad 1 < x \le 2. \end{cases}$$

The Complex Fourier series of f^* on the interval $[-1, 1]$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x}, \tag{1}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx = \int_{-1}^{0} e^{-in\pi x}dx \\ &= \Big[\frac{1}{-in\pi}e^{-in\pi x}\Big]_{-1}^{0} \\ &= \frac{1}{in\pi}(e^{in\pi} - 1) \\ &= \frac{i}{n\pi}(1 - (-1)^n). \end{aligned} \tag{2}$$

Using (1) and (2), we get the Complex Fourier series of f^* on the interval $[-1, 1]$

$$\sum_{n=-\infty}^{\infty} \frac{i}{n\pi}(1-(-1)^n)e^{in\pi x}.$$

The Complex Fourier series of f on the interval $[0, 2]$ is given by

$$\begin{array}{ll} \displaystyle\sum_{n=-\infty}^{\infty} \frac{i}{n\pi}(1-(-1)^n)e^{in\pi x} & \text{if} \quad 0 \le x \le 1 \\ \displaystyle\sum_{n=-\infty}^{\infty} \frac{i}{n\pi}(1-(-1)^n)e^{in\pi(x-2)} = \sum_{n=-\infty}^{\infty} \frac{i}{n\pi}(1-(-1)^n)e^{in\pi x} & \text{if} \quad 1 < x \le 2. \end{array}$$

Hence the Complex Fourier series of f on the interval $[0, 2]$ is given by

$$\sum_{n=-\infty}^{\infty} \frac{i}{n\pi}(1-(-1)^n)e^{in\pi x}.$$

□

4. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad -\pi \le x < 0 \\ x, & \text{if} \quad 0 \le x \le \pi. \end{cases}$$

The Complex Fourier series of f on the interval $[-\pi, \pi]$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{\pi}x},$$

or

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{\frac{-in\pi}{\pi}x}dx = \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}dx. \tag{2}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_0^{\pi} xe^{-inx}dx &= \left[x\frac{1}{-in}e^{-inx}\right]_0^{\pi} - \int_0^{\pi} \frac{1}{-in}e^{-inx}dx \\ &= \frac{\pi}{-in}e^{-in\pi} + \frac{1}{in}\left[\frac{1}{-in}e^{-inx}\right]_0^{\pi} \\ &= \frac{\pi}{-in}e^{-in\pi} + \frac{1}{n^2}(e^{-in\pi} - 1) \\ &= i\pi\frac{(-1)^n}{n} + \frac{1}{n^2}((-1)^n - 1). \end{aligned} \tag{3}$$

Using (2) and (3), we get

$$c_n = i\frac{(-1)^n}{2n} + \frac{(-1)^n - 1}{2\pi n^2}. \tag{4}$$

Using (1) and (4), we get the Complex Fourier series of f on the interval $[-\pi, \pi]$

$$\sum_{n=-\infty}^{\infty} \left(i\frac{(-1)^n}{2n} + \frac{(-1)^n - 1}{2\pi n^2}\right)e^{inx}.$$

□

6. Let f be the function defined by $f(x) = e^{-|x|}$
The Complex Fourier series of f on the interval $[-1, 1]$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{1}x},$$

or

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x}, \tag{1}$$

where

$$
\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{\frac{-in\pi}{1}x}dx = \frac{1}{2}\int_{-1}^{0} e^{x}e^{-in\pi x}dx + \frac{1}{2}\int_{0}^{1} e^{-x}e^{-in\pi x}dx \\
&= \frac{1}{2}\int_{-1}^{0} e^{(1-in\pi)x}dx + \frac{1}{2}\int_{0}^{1} e^{-(1+in\pi)x}dx \\
&= \frac{1}{2}\Big[\frac{1}{1-in\pi}e^{(1-in\pi)x}\Big]_{-1}^{0} + \frac{1}{2}\Big[\frac{1}{-1-in\pi}e^{-(1+in\pi)x}\Big]_{0}^{1} \\
&= \frac{1}{2}\Big(\frac{1}{1-in\pi} - \frac{e^{(-1+in\pi)}}{1-in\pi} + \frac{e^{-(1+in\pi)}}{-1-in\pi} + \frac{1}{1+in\pi}\Big) \\
&= \frac{1}{2}\Big(\frac{1}{1-in\pi} - \frac{(-1)^n e^{-1}}{1-in\pi} - \frac{(-1)^n e^{-1}}{1+in\pi} + \frac{1}{1+in\pi}\Big) \\
&= \frac{1}{2}(1-(-1)^n e^{-1})\Big(\frac{1}{1-in\pi} + \frac{1}{1+in\pi}\Big) \\
&= \frac{1-(-1)^n e^{-1}}{1+n^2\pi^2}. \qquad (2)
\end{aligned}
$$

Using (1) and (4), we get the Complex Fourier series of f on the interval $[-1, 1]$

$$\sum_{n=-\infty}^{\infty} \frac{1-(-1)^n e^{-1}}{1+n^2\pi^2} e^{in\pi x}.$$

□