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6. Let $f_1(x) = e^x$ and $f_2(x) = \sin x$. We would like to show that f_1 and f_2 are orthogonal on $[\pi/4, 5\pi/4]$. Indeed using a double integration by parts, we get

$$
\begin{aligned}
(f_1, f_2) = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x)e^x dx &= \big[\sin(x)e^x\big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)e^x dx \\
&= \frac{\sqrt{2}}{2}\big(e^{\frac{5\pi}{4}} - e^{\frac{\pi}{4}}\big) - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)e^x dx \\
&= \frac{\sqrt{2}}{2}\big(e^{\frac{5\pi}{4}} - e^{\frac{\pi}{4}}\big) - \Big(\big[\cos(x)e^x\big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} -\sin(x)e^x dx\Big) \\
&= \frac{\sqrt{2}}{2}\big(e^{\frac{5\pi}{4}} - e^{\frac{\pi}{4}}\big) - \frac{\sqrt{2}}{2}\big(e^{\frac{5\pi}{4}} - e^{\frac{\pi}{4}}\big) - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x)e^x dx \\
&= -(f_1, f_2).
\end{aligned}
$$

It follows that $2(f_1, f_2) = 0$, which leads to $(f_1, f_2) = 0$. □

12. We would like to show that the set of functions $\{\phi_0, \phi_n, \psi_m\}$, $n, m = 1, 2, 3, ...$ where $\phi_0(x) = 1$, $\phi_n(x) = \cos\left(\frac{n\pi}{p}x\right)$, $\psi_m(x) = \sin\left(\frac{m\pi}{p}x\right)$ are orthogonal on $[-p, p]$. Indeed we have first for $n, m = 1, 2, 3, ...$

$$
\begin{aligned}
(\phi_0, \phi_n) = \int_{-p}^{p} \cos\Big(\frac{n\pi}{p}x\Big)dx &= \Big[\frac{p}{n\pi}\sin\Big(\frac{n\pi}{p}x\Big)\Big]_{-p}^{p} \\
&= \frac{p}{n\pi}\sin(n\pi) - \frac{p}{n\pi}\sin(-n\pi) \\
&= 0 - 0 = 0.
\end{aligned}
$$

$$
\begin{aligned}
(\phi_0, \psi_m) = \int_{-p}^{p} \sin\Big(\frac{m\pi}{p}x\Big)dx &= \Big[-\frac{p}{m\pi}\cos\Big(\frac{m\pi}{p}x\Big)\Big]_{-p}^{p} \\
&= -\frac{p}{m\pi}\cos(m\pi) + \frac{p}{m\pi}\cos(-m\pi) \\
&= -\frac{p}{m\pi}\cos(m\pi) + \frac{p}{m\pi}\cos(m\pi) \\
&= 0.
\end{aligned}
$$

For $n, m = 1, 2, 3, ...$ with $n \neq m$

$$
\begin{aligned}
(\phi_n, \psi_m) &= \int_{-p}^{p} \cos\Big(\frac{n\pi}{p}x\Big)\sin\Big(\frac{m\pi}{p}x\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\Big(\sin\Big(\frac{(m+n)\pi}{p}x\Big) + \sin\Big(\frac{(m-n)\pi}{p}x\Big)\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\sin\Big(\frac{(m+n)\pi}{p}x\Big)dx + \frac{1}{2}\int_{-p}^{p}\sin\Big(\frac{(m-n)\pi}{p}x\Big)dx \\
&= \frac{1}{2}(\phi_0, \psi_{m+n}) + \frac{1}{2}(\phi_0, \psi_{m-n}) \\
&= 0 + 0 \\
&= 0.
\end{aligned}
$$

For $n, m = 1, 2, 3, ...$ with $n = m$

$$
\begin{aligned}
(\phi_n, \psi_m) &= \int_{-p}^{p} \cos\Big(\frac{n\pi}{p}x\Big)\sin\Big(\frac{n\pi}{p}x\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\sin\Big(\frac{2n\pi}{p}x\Big)dx \\
&= \frac{1}{2}(\phi_0, \psi_{2n}) \\
&= 0.
\end{aligned}
$$

Now we have for $n, m = 1, 2, 3, ...$ with $n \neq m$

$$\begin{aligned}
(\phi_n, \phi_m) &= \int_{-p}^{p} \cos\Big(\frac{n\pi}{p}x\Big)\cos\Big(\frac{m\pi}{p}x\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\Big(\cos\Big(\frac{(m+n)\pi}{p}x\Big) + \cos\Big(\frac{(m-n)\pi}{p}x\Big)\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\cos\Big(\frac{(m+n)\pi}{p}x\Big)dx + \frac{1}{2}\int_{-p}^{p}\cos\Big(\frac{(m-n)\pi}{p}x\Big)dx \\
&= \frac{1}{2}(\phi_0, \phi_{m+n}) + \frac{1}{2}(\phi_0, \phi_{m-n}) \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
(\psi_n, \psi_m) &= \int_{-p}^{p} \sin\Big(\frac{n\pi}{p}x\Big)\sin\Big(\frac{m\pi}{p}x\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\Big(\cos\Big(\frac{(m-n)\pi}{p}x\Big) - \cos\Big(\frac{(m+n)\pi}{p}x\Big)\Big)dx \\
&= \frac{1}{2}\int_{-p}^{p}\cos\Big(\frac{(m-n)\pi}{p}x\Big)dx - \frac{1}{2}\int_{-p}^{p}\cos\Big(\frac{(m+n)\pi}{p}x\Big)dx \\
&= \frac{1}{2}(\phi_0, \phi_{m-n}) - \frac{1}{2}(\phi_0, \phi_{m+n}) \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

Let us now evaluate the norms of the functions ϕ_0, ϕ_n, ψ_m

$$||\phi_0||^2 = \int_{-p}^{p} 1^2 dx = 2p.$$

$$
\begin{aligned}
||\phi_n||^2 &= \int_{-p}^{p} \cos^2\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{1}{2}\int_{-p}^{p}\left(1+\cos\left(\frac{2n\pi}{p}x\right)dx\right. \\
&= \frac{1}{2}\int_{-p}^{p} dx + \frac{1}{2}\int_{-p}^{p}\cos\left(\frac{2n\pi}{p}x\right)dx \\
&= \frac{1}{2}(2p) + \frac{1}{2}(\phi_0, \phi_{2n}) \\
&= p + 0 = p.
\end{aligned}
$$

$$
\begin{aligned}
||\psi_n||^2 &= \int_{-p}^{p} \sin^2\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{1}{2}\int_{-p}^{p}\left(1-\cos\left(\frac{2n\pi}{p}x\right)dx\right. \\
&= \frac{1}{2}\int_{-p}^{p} dx - \frac{1}{2}\int_{-p}^{p}\cos\left(\frac{2n\pi}{p}x\right)dx \\
&= \frac{1}{2}(2p) - \frac{1}{2}(\phi_0, \phi_{2n}) \\
&= p - 0 = p.
\end{aligned}
$$

Hence $||\phi_0|| = \sqrt{2p}$, $||\phi_n|| = \sqrt{p}$ and $||\psi_n|| = \sqrt{p}$. □

16. Let $\{\phi_n\}$, $n = 0, 1, 2, 3, ...$ be a set of orthogonal functions on $[a, b]$, where $\phi_0(x) = 1$ and $\phi_1(x) = x$. We would like to show that for any constants α, β, we have

$$\int_a^b (\alpha x + \beta)\phi_n(x)dx = 0 \quad \text{for all} \quad n = 2, 3, ...$$

Indeed we have for any α, β and any $n = 2, 3, ...$

$$\begin{aligned}
\int_a^b (\alpha x+\beta)\phi_n(x)dx &= \int_a^b (\alpha\phi_1(x)+\beta\phi_0(x))\phi_n(x)dx \\
&= \alpha\int_a^b \phi_1(x)\phi_n(x)dx + \beta\int_a^b \phi_0(x)\phi_n(x)dx \\
&= \\
&= \alpha(\phi_1,\phi_n)+\beta(\phi_0,\phi_n) \\
&= \alpha 0+\beta 0 = 0.
\end{aligned}$$

□

18. From Pb 1, we know that $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal functions on $[-2, 2]$. We want to find c_1 and c_2 such that $f_3(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f_1 and f_2 on $[-2, 2]$.

Let $f_4(x) = x^3$. We remark that $f_3 = f_1 + c_1f_2 + c_2f_4$. Then we have

$$\begin{aligned}
(f_3, f_1) &= (f_1 + c_1f_2 + c_2f_4, f_1) \\
&= (f_1, f_1) + c_1(f_2, f_1) + c_2(f_4, f_1) \\
&= (f_1, f_1) + c_2(f_4, f_1). \qquad (1)
\end{aligned}$$

$$\begin{aligned}
(f_1, f_1) &= \int_{-2}^2 x^2 dx \\
&= \left[\frac{x^3}{3}\right]_{-2}^2 \\
&= \frac{2^3}{3} - \frac{(-2)^3}{3} = \frac{16}{3}. \qquad (2)
\end{aligned}$$

$$\begin{aligned}
(f_4, f_1) &= \int_{-2}^2 x^4 dx \\
&= \left[\frac{x^5}{5}\right]_{-2}^2 \\
&= \frac{2^5}{5} - \frac{(-2)^5}{5} = \frac{64}{5}. \qquad (3)
\end{aligned}$$

Similarly we have

$$\begin{aligned}(f_3, f_2) &= (f_1 + c_1 f_2 + c_2 f_4, f_2) \\ &= (f_1, f_2) + c_1(f_2, f_2) + c_2(f_4, f_2) \\ &= c_1(f_2, f_2) + c_2(f_4, f_2). \end{aligned} \tag{4}$$

$$\begin{aligned}(f_2, f_2) &= \int_{-2}^{2} x^4 dx \\ &= \left[\frac{x^5}{5}\right]_{-2}^{2} \\ &= \frac{2^5}{5} - \frac{(-2)^5}{5} = \frac{64}{5}. \end{aligned} \tag{5}$$

$$\begin{aligned}(f_4, f_2) &= \int_{-2}^{2} x^5 dx \\ &= \left[\frac{x^6}{6}\right]_{-2}^{2} \\ &= \frac{2^6}{6} - \frac{(-2)^6}{6} = 0. \end{aligned} \tag{6}$$

Now f_3 is orthogonal to both f_1 and f_2 on $[-2, 2]$ if and only if $(f_3, f_1) = 0$ and $(f_3, f_2) = 0$, which is equivalent by (1), (2), (3) and (4), to the linear system:

$$\left\{ \begin{array}{l} \frac{16}{3} + \frac{64}{5}c_2 = 0 \\ \frac{64}{5}c_1 = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} c_2 = -\frac{5}{12} \\ c_1 = 0, \end{array} \right. \quad \Leftrightarrow \quad f_3(x) = x - \frac{5}{12}x^3.$$

□