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4. We consider the initial-value problem

$$\begin{cases} y'' + 16y = \delta_{2\pi}(t), \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$\begin{aligned} & s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \mathcal{L}(\delta_{2\pi}) \\ & \quad \Leftrightarrow \ (s^2+16)Y(s) = e^{-2\pi s} \\ & \quad \Leftrightarrow \ Y(s) = e^{-2\pi s}\frac{1}{s^2+16} = \frac{1}{4}e^{-2\pi s}\frac{4}{s^2+16} = \frac{1}{4}e^{-2\pi s}F(s), \end{aligned} \tag{1}$$

where $F(s) = \mathcal{L}(\sin(4t))$. Using the formula

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \ \text{with } a = 2\pi, \ f(t) = \sin(4t)$$

we get from (1)

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{4}\sin(4(t-2\pi))\mathcal{U}_{2\pi}(t) = \frac{1}{4}\sin(4t)\mathcal{U}_{2\pi}(t).$$

$\square$

8. We consider the initial-value problem

$$\begin{cases} y'' - 2y' = 1 + \delta_2(t), \\ y(0) = 0, \ y'(0) = 1. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$\begin{aligned}
& s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) = \mathcal{L}(1) + \mathcal{L}(\delta_2) \\
& \quad \Leftrightarrow \; (s^2 - 2s)Y(s) = 1 + \frac{1}{s} + e^{-2s} \\
& \quad \Leftrightarrow \; Y(s) = \frac{1}{s(s-2)} + \frac{1}{s^2(s-2)} + e^{-2s}\frac{1}{s(s-2)} \\
& \quad \Leftrightarrow \; Y(s) = \frac{s+1}{s^2(s-2)} + e^{-2s}\frac{1}{s(s-2)}.
\end{aligned} \tag{1}$$

Note that

$$\frac{s+1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} = \frac{(A+C)s^2 + (B-2A)s - 2B}{s^2(s-2)}. \tag{2}$$

We deduce from (2) that

$$\begin{cases} A + C = 0, \\ B - 2A = 1, \\ -2B = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = -3/4, \\ B = -1/2, \\ C = 3/4 \end{cases}$$

We also have

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} = \frac{(A+B)s - 2A}{s(s-2)}. \tag{3}$$

It follows from (3) that

$$\begin{cases} A + B = 0, \\ -2A = 1, \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = -1/2, \\ B = 1/2, \end{cases}$$

We deduce from (1), (2) and (3) that

$$Y(s) = -\frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s-2} - \frac{1}{2}e^{-2s}\frac{1}{s} + \frac{1}{2}e^{-2s}\frac{1}{s-2}. \tag{4}$$

Using the formula

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a = 2, \; f(t) = 1 \text{ and } f(t) = e^{2t},$$

we get from (4)

$$y(t)=\mathcal{L}^{-1}(Y(s))=-\frac{3}{4}-\frac{1}{2}t+\frac{3}{4}e^{2t}-\frac{1}{2}\mathcal{U}_2(t)+\frac{1}{2}e^{2(t-2)}\mathcal{U}_2(t).$$

$\square$

12. We consider the initial-value problem

$$\begin{cases} y''-7y'+6y=e^t+\delta_2(t)+\delta_4(t), \\ y(0)=0,\ y'(0)=0. \end{cases}$$

Let $Y(s)=\mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$\begin{aligned} & s^2Y(s)-sy(0)-y'(0)-7(sY(s)-y(0))+6Y(s)=\mathcal{L}(e^t)+\mathcal{L}(\delta_2)+\mathcal{L}(\delta_4) \\ & \Leftrightarrow\ (s^2-7s+6)Y(s)=\frac{1}{s-1}+e^{-2s}+e^{-4s} \\ & \Leftrightarrow\ Y(s)=\frac{1}{(s-1)(s^2-7s+6)}+e^{-2s}\frac{1}{(s^2-7s+6)}+e^{-4s}\frac{1}{(s^2-7s+6)} \\ & \Leftrightarrow\ Y(s)=\frac{1}{(s-1)^2(s-6)}+e^{-2s}\frac{1}{(s-1)(s-6)}+e^{-4s}\frac{1}{(s-1)(s-6)}. \qquad (1) \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{(s-1)^2(s-6)} &= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s-6} \\ &= \frac{(A+C)s^2+(-7A+B-2C)s+6A-6B+C}{(s-1)^2(s-6)}. \qquad (2) \end{aligned}$$

We deduce from (2) that

$$\begin{cases} A+C=0, \\ -7A+B-2C=0, \\ 6A-6B+C=1 \end{cases} \Leftrightarrow \begin{cases} C=-A, \\ -5A+B=0, \\ 5A-6B=1 \end{cases} \Leftrightarrow \begin{cases} A=-1/25, \\ B=-1/5, \\ C=1/25 \end{cases}$$

We also have

$$\frac{1}{(s-1)(s-6)} = \frac{A}{s-1} + \frac{B}{s-6} = \frac{(A+B)s - 6A - B}{(s-1)(s-6)}. \tag{3}$$

It follows from (3) that

$$\begin{cases} A + B = 0, \\ -6A - B = 1, \end{cases} \qquad \Leftrightarrow \quad \begin{cases} A = -1/5, \\ B = 1/5, \end{cases}$$

We deduce from (1), (2) and (3) that

$$\begin{aligned} Y(s) = & -\frac{1}{25}\frac{1}{s-1} - \frac{1}{5}\frac{1}{(s-1)^2} + \frac{1}{25}\frac{1}{s-6} \\ & -\frac{1}{5}e^{-2s}\frac{1}{s-1} + \frac{1}{5}e^{-2s}\frac{1}{s-6} - \frac{1}{5}e^{-4s}\frac{1}{s-1} + \frac{1}{5}e^{-4s}\frac{1}{s-6}. \end{aligned} \tag{4}$$

Using the formulas

$$\begin{aligned} &\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t), \text{ with } a = 1, \ f(t) = t \text{ and } F(s) = \frac{1}{s^2} \\ &\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a = 2, 4 \text{ and } f(t) = e^t, e^{6t} \end{aligned}$$

we get from (4)

$$\begin{aligned} y(t) = \mathcal{L}^{-1}(Y(s)) = & -\frac{1}{25}e^t - \frac{1}{5}te^t + \frac{1}{25}e^{6t} - \frac{1}{5}e^{t-2}\mathcal{U}_2(t) \\ & + \frac{1}{5}e^{6(t-2)}\mathcal{U}_2(t) - \frac{1}{5}e^{t-4}\mathcal{U}_4(t) + \frac{1}{5}e^{6(t-4)}\mathcal{U}_4(t). \end{aligned}$$

$\square$