King Fahd University of Petroleum and Minerals

Department of Mathematical Sciences

Dr. A. Lyaghfouri

MATH 301/Term 062/Hw#12(4.3)/

8. We would like to evaluate $\mathcal{L}(e^{-2t}\cos(4t))$. We will use the formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a), \text{ with } a=-2, f(t)=\cos(4t), \text{ and } F(s)=\frac{s}{s^2+16}.$$

Hence we get

$$\mathcal{L}(e^{-2t}\cos(4t)) = \frac{s+2}{(s+2)^2+16}.$$

□

13. We would like to evaluate $\mathcal{L}^{-1}\Big(\frac{1}{s^2-6s+10}\Big)$. Note that

$$\frac{1}{s^2-6s+10} = \frac{1}{(s-3)^2+1} = F(s-3)$$

where $F(s) = \frac{1}{s^2+1} = \mathcal{L}(\sin(t))$. Using the formula

$$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t), \text{ with } a=3, f(t)=\sin(t), \text{ and } F(s)=\frac{1}{s^2+1},$$

we get

$$\mathcal{L}^{-1}\Big(\frac{1}{s^2-6s+10}\Big) = e^{3t}\sin(t).$$

□

20. We would like to evaluate $\mathcal{L}^{-1}\Big(\frac{(s+1)^2}{(s+2)^4}\Big)$. Note that

$$
\begin{aligned}
\frac{(s+1)^2}{(s+2)^4} &= \frac{(s+2-1)^2}{(s+2)^4} = \frac{(s+2)^2 - 2(s+2) + 1}{(s+2)^4} \\
&= \frac{1}{(s+2)^2} - 2\frac{1}{(s+2)^3} + \frac{1}{(s+2)^4} \\
&= F_1(s+2) - 2F_2(s+2) + F_3(s+2)
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(s) &= \frac{1}{s^2} = \mathcal{L}(t) \\
F_2(s) &= \frac{1}{s^3} = \mathcal{L}\Big(\frac{t^2}{2}\Big) \\
F_3(s) &= \frac{1}{s^4} = \mathcal{L}\Big(\frac{t^3}{6}\Big)
\end{aligned}
$$

Using the formula

$$
\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t), \text{ with } a = -2,
$$

we get

$$
\begin{aligned}
\mathcal{L}^{-1}\Big(\frac{(s+1)^2}{(s+2)^4}\Big) &= \mathcal{L}^{-1}(F_1(s+2)) - 2\mathcal{L}^{-1}(F_2(s+2)) + \mathcal{L}^{-1}(F_3(s+2)) \\
&= e^{-2t} t - 2e^{-2t}\frac{t^2}{2} + e^{-2t}\frac{t^3}{6} \\
&= e^{-2t}\Big(t - t^2 + \frac{t^3}{6}\Big).
\end{aligned}
$$

□

24. We consider the initial-value problem

$$
\begin{cases} y'' - 4y' + 4y = t^3 e^{2t}, \\ y(0) = 0, \ y'(0) = 0. \end{cases}
$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$
\begin{aligned}
& s^2 Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \mathcal{L}(t^3 e^{2t}) \\
\Leftrightarrow\ & (s^2 - 4s + 4)Y(s) = \mathcal{L}(t^3)(s-2) = \frac{6}{(s-2)^4} \\
\Leftrightarrow\ & Y(s) = \frac{6}{(s-2)^6} = \frac{6}{5!}F(s-2), \text{ where } F(s) = \mathcal{L}(t^5). \qquad (1)
\end{aligned}
$$

Using the formula

$$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t), \text{ with } a = 2,$$

we get from (1)

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{6}{5!}t^5e^{2t} = \frac{1}{20}t^5e^{2t}.$$

□

38. We would like to evaluate $\mathcal{L}(e^{2-t}\mathcal{U}_2(t))$. We will use the formula

$$\mathcal{L}(f(t-a)\mathcal{U}_a(t)) = e^{-as}F(s), \text{ with } a = 2, f(t) = e^{-t}, \text{ and } F(s) = \frac{1}{s+1}.$$

Hence we get

$$\mathcal{L}(e^{2-t}\mathcal{U}_2(t)) = e^{-2s}\frac{1}{s+1} = \frac{e^{-2s}}{s+1}.$$

□

47. We would like to evaluate $\mathcal{L}^{-1}\Big(\frac{e^{-s}}{s(s+1)}\Big)$.
Note that

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{(A+B)s+A}{s(s+1)}. \tag{1}$$

We deduce from (1) that

$$\begin{cases} A+B=0, \\ A=1 \end{cases} \Leftrightarrow \begin{cases} A=1, \\ B=-1 \end{cases}$$

It follows from (1) that

$$\frac{e^{-s}}{s(s+1)} = e^{-s}\frac{1}{s} - e^{-s}\frac{1}{s+1}$$

which leads to

$$\begin{aligned}
\mathcal{L}^{-1}\Big(\frac{e^{-s}}{s(s+1)}\Big) &= \mathcal{L}^{-1}\Big(e^{-s}\frac{1}{s}\Big) - \mathcal{L}^{-1}\Big(e^{-s}\frac{1}{s+1}\Big)\\
&= \mathcal{L}^{-1}(e^{-s}F_1(s)) - \mathcal{L}^{-1}(e^{-s}F_2(s))
\end{aligned}$$

where

$$\begin{aligned}
F_1(s) &= \frac{1}{s} = \mathcal{L}(1)\\
F_2(s) &= \frac{1}{s+1} = \mathcal{L}(e^{-t})
\end{aligned}$$

Using the formula

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a = 1,$$

we get

$$\mathcal{L}^{-1}\Big(\frac{e^{-s}}{s(s+1)}\Big) = \mathcal{U}_1(t) - e^{-(t-1)}\mathcal{U}_1(t) = (1 - e^{1-t})\mathcal{U}_1(t).$$

□

66. We consider the initial-value problem

$$\begin{cases} y'' + 4y = f(t),\\ y(0) = 0,\ y'(0) = -1 \end{cases}$$

where

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1\\ 0, & \text{if } t \ge 1. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions and the fact that $f(t) = 1 - \mathcal{U}_1(t)$, we get

$$\begin{aligned}
& s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \mathcal{L}(1) - \mathcal{L}(\mathcal{U}_1(t)) = \frac{1}{s} - \frac{e^{-s}}{s}\\
&\Leftrightarrow\ (s^2+4)Y(s) + 1 = \frac{1}{s} - \frac{e^{-s}}{s}\\
&\Leftrightarrow\ Y(s) = -\frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)}. \qquad (1)
\end{aligned}$$

Note that

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2+Cs+4A}{s(s^2+4)}. \tag{2}$$

We deduce from (2) that

$$\begin{cases} 4A = 1, \\ A+B=0, \\ C=0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = 1/4, \\ B=-1/4, \\ C=0 \end{cases}$$

It follows from (1) and (2) that

$$Y(s) = -\frac{1}{s^2+4} + \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{4}e^{-s}\frac{1}{s} + \frac{1}{4}e^{-s}\frac{s}{s^2+4}$$

which leads to

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y(s)) = -\frac{1}{2}\mathcal{L}^{-1}\Big(\frac{2}{s^2+4}\Big) + \frac{1}{4}\mathcal{L}^{-1}\Big(\frac{1}{s}\Big) - \frac{1}{4}\mathcal{L}^{-1}\Big(e^{-s}\frac{1}{s}\Big) + \frac{1}{4}\mathcal{L}^{-1}\Big(e^{-s}\frac{s}{s^2+4}\Big) \\ &= -\frac{1}{2}\sin(2t) + \frac{1}{4} - \frac{1}{4}\mathcal{L}^{-1}(e^{-s}F_1(s)) + \frac{1}{4}\mathcal{L}^{-1}(e^{-s}F_2(s)) \end{aligned}$$

where

$$\begin{aligned} F_1(s) &= \frac{1}{s} = \mathcal{L}(1) \\ F_2(s) &= \frac{s}{s^2+4} = \mathcal{L}(\cos(2t)). \end{aligned}$$

Using the formula

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a=1,$$

we get

$$y(t) = -\frac{1}{2}\sin(2t) + \frac{1}{4} - \frac{1}{4}\mathcal{U}_1(t) + \frac{1}{4}\cos(2(t-1))\mathcal{U}_1(t).$$

□