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MATH 301/Term 062/Hw#11(4.2)/

5. Let $F(s) = \dfrac{(s+1)^3}{s^4}$. Note that

$$F(s) = \frac{s^3 + 3s^2 + 3s + 1}{s^4} = \frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}.$$

Then we have

$$\begin{aligned}
\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}\Big(\frac{1}{s}\Big) + \mathcal{L}^{-1}\Big(\frac{3}{s^2}\Big) + \mathcal{L}^{-1}\Big(\frac{3}{s^3}\Big) + \mathcal{L}^{-1}\Big(\frac{1}{s^4}\Big) \\
&= \mathcal{L}^{-1}\Big(\frac{1}{s}\Big) + 3\mathcal{L}^{-1}\Big(\frac{1!}{s^2}\Big) + \frac{3}{2}\mathcal{L}^{-1}\Big(\frac{2!}{s^3}\Big) + \frac{1}{3!}\mathcal{L}^{-1}\Big(\frac{3!}{s^4}\Big) \\
&= 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3.
\end{aligned}$$

□

12. Let $F(s) = \dfrac{10s}{s^2+16}$. We have

$$\mathcal{L}^{-1}(F(s)) = 10\mathcal{L}^{-1}\Big(\frac{s}{s^2+4^2}\Big) = 10\cos(4t).$$

□

16. Let $F(s) = \dfrac{s+1}{s^2+2}$. Note that

$$F(s) = \frac{s}{s^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2}.$$

Then we have

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}\Big(\frac{s}{s^2+2}\Big) + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\Big(\frac{s}{s^2+2}\Big)\\ &= \cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t).\end{aligned}$$

□

35. We consider the initial-value problem

$$\begin{cases} y'' + 5y' + 4y = 0, \\ y(0) = 1,\ y'(0) = 0. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$\begin{aligned} & s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 4Y(s) = 0 \\ \Leftrightarrow\ & (s^2+5s+4)Y(s) - s - 5 = 0 \\ \Leftrightarrow\ & Y(s) = \frac{s+5}{s^2+5s+4}. \end{aligned} \tag{1}$$

Note that

$$\begin{aligned}\frac{s+5}{s^2+5s+4} &= \frac{s+5}{(s+5/2)^2-(3/2)^2} = \frac{s+5}{(s+1)(s+4)} \\ &= \frac{A}{s+1} + \frac{B}{s+4} = \frac{(A+B)s+4A+B}{(s+1)(s+4)}.\end{aligned} \tag{2}$$

We deduce from (2) that

$$\begin{cases} A+B=1, \\ 4A+B=5 \end{cases} \Leftrightarrow \begin{cases} 3A=4, \\ B=1-A \end{cases} \Leftrightarrow \begin{cases} A=4/3, \\ B=-1/3. \end{cases}$$

It follows from (1) and (2) that

$$Y(s) = \frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}$$

which leads to

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}\mathcal{L}^{-1}\Big(\frac{1}{s+1}\Big) - \frac{1}{3}\mathcal{L}^{-1}\Big(\frac{1}{s+4}\Big) \\ &= \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}. \end{aligned}$$

□

37. We consider the initial-value problem

$$\begin{cases} y'' + y = \sqrt{2}\sin(\sqrt{2}t), \\ y(0) = 10, \ y'(0) = 0. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$\begin{aligned} & s^2Y(s) - sy(0) - y'(0) + Y(s) = \sqrt{2}\frac{\sqrt{2}}{s^2+2} \\ \Leftrightarrow\ & (s^2+1)Y(s) - 10s = \frac{2}{s^2+2} \\ \Leftrightarrow\ & Y(s) = \frac{10s}{s^2+1} + \frac{2}{(s^2+1)(s^2+2)}. \end{aligned} \tag{1}$$

Note that

$$\frac{2}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{(A+B)x + 2A + B}{(x+1)(x+2)}. \tag{2}$$

We deduce from (2) that

$$\begin{cases} A + B = 0, \\ 2A + B = 2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = 2, \\ B = -A = -2 \end{cases}$$

It follows from (1) and (2) that

$$Y(s) = 10\frac{s}{s^2+1} + 2\frac{1}{s^2+1} - \sqrt{2}\frac{\sqrt{2}}{s^2+2}$$

which leads to

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}(Y(s)) = 10\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \sqrt{2}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2+2}\right) \\
&= 10\cos(t) + 2\sin(t) - \sqrt{2}\sin(\sqrt{2}t).
\end{aligned}
$$

□