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MATH 301/Term 062/Hw#10(4.1)/

4. Let $f(t)$ be the function defined by

$$f(t) = \begin{cases} 2t+1, & \text{if } \quad 0 \le t < 1 \\ 0, & \text{if } \quad t \ge 1. \end{cases}$$

The Laplace transform of f is given by

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)dt = \int_0^1 (2t+1)e^{-st}dt \\
&= \Big[(2t+1)\frac{e^{-st}}{-s}\Big]_0^1 - \int_0^1 2\frac{e^{-st}}{-s}dt \\
&= \Big[3\frac{e^{-s}}{-s} + \frac{1}{s}\Big] + \frac{2}{s}\int_0^1 e^{-st}dt \\
&= \frac{1-3e^{-s}}{s} + \frac{2}{s}\Big[\frac{e^{-st}}{-s}\Big]_0^1 \\
&= \frac{1-3e^{-s}}{s} + \frac{2}{s}\frac{1-e^{-s}}{s} \\
&= \frac{1-3e^{-s}}{s} + \frac{2(1-e^{-s})}{s^2}.
\end{aligned}$$

□

6. Let $f(t)$ be the function defined by

$$f(t) = \begin{cases} \sin t, & \text{if } \quad 0 \le t < \pi/2 \\ 0, & \text{if } \quad t \ge \pi/2. \end{cases}$$

The Laplace transform of f is given by

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = \int_0^{\frac{\pi}{2}} \sin(t)e^{-st}dt. \tag{1}$$

Using a double integration by parts, we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin(t)e^{-st}dt &= \left[\sin(t)\frac{e^{-st}}{-s}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(t)\frac{e^{-st}}{-s}dt \\
&= -\frac{e^{-s\frac{\pi}{2}}}{s} + \frac{1}{s}\int_0^{\frac{\pi}{2}} \cos(t)e^{-st}dt \\
&= -\frac{e^{-s\frac{\pi}{2}}}{s} + \frac{1}{s}\left[\cos(t)\frac{e^{-st}}{-s}\right]_0^{\frac{\pi}{2}} - \frac{1}{s}\int_0^{\frac{\pi}{2}} -\sin(t)\frac{e^{-st}}{-s}dt \\
&= -\frac{e^{-s\frac{\pi}{2}}}{s} + \frac{1}{s^2} - \frac{1}{s^2}\int_0^{\frac{\pi}{2}} \sin(t)e^{-st}dt. \qquad (2)
\end{aligned}$$

We deduce from (2) that

$$\left(1+\frac{1}{s^2}\right)\int_0^{\frac{\pi}{2}} \sin(t)e^{-st}dt = \frac{1}{s^2} - \frac{e^{-s\frac{\pi}{2}}}{s}$$

which leads to

$$\int_0^{\frac{\pi}{2}} \sin(t)e^{-st}dt = \frac{1}{1+s^2} - \frac{se^{-s\frac{\pi}{2}}}{1+s^2}. \qquad (3)$$

Hence we obtain from (1) and (3)

$$\mathcal{L}(f(t)) = \frac{1}{1+s^2} - \frac{se^{-s\frac{\pi}{2}}}{1+s^2}.$$

□

18. Let Let $f(t) = t\sin(t)$. The Laplace transform of f is given by

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st}f(t)dt = \int_0^{\infty} t\sin(t)e^{-st}dt. \qquad (1)$$

Let us first evaluate, by integrating by parts, the integral

$$
\begin{aligned}
\int_0^t \sin(\tau)e^{-s\tau}d\tau &= \Big[\sin(\tau)\frac{e^{-s\tau}}{-s}\Big]_0^t - \int_0^t \cos(\tau)\frac{e^{-s\tau}}{-s}d\tau \\
&= -\sin(t)\frac{e^{-st}}{s} + \frac{1}{s}\int_0^t \cos(\tau)e^{-s\tau}d\tau \qquad (2)\\
&= -\sin(t)\frac{e^{-st}}{s} + \frac{1}{s}\Big[\cos(\tau)\frac{e^{-s\tau}}{-s}\Big]_0^t - \frac{1}{s}\int_0^t -\sin(\tau)\frac{e^{-s\tau}}{-s}d\tau \\
&= -\sin(t)\frac{e^{-st}}{s} - \cos(t)\frac{e^{-st}}{s^2} + \frac{1}{s^2} - \frac{1}{s^2}\int_0^t \sin(\tau)e^{-s\tau}d\tau. \qquad (3)
\end{aligned}
$$

We deduce from (3) that

$$
\Big(1+\frac{1}{s^2}\Big)\int_0^t \sin(\tau)e^{-s\tau}d\tau = \frac{1}{s^2} - \sin(t)\frac{e^{-st}}{s} - \cos(t)\frac{e^{-st}}{s^2}
$$

which leads to

$$
\int_0^t \sin(\tau)e^{-s\tau}d\tau = \frac{1}{1+s^2} - \sin(t)\frac{se^{-st}}{1+s^2} - \cos(t)\frac{e^{-st}}{1+s^2}. \qquad (4)
$$

We also deduce from (2) that

$$
\begin{aligned}
\int_0^t \cos(\tau)e^{-s\tau}d\tau &= \sin(t)e^{-st} + s\int_0^t \sin(\tau)e^{-s\tau}d\tau \\
&= \sin(t)e^{-st} + \frac{s}{1+s^2} - \sin(t)\frac{s^2e^{-st}}{1+s^2} - \cos(t)\frac{se^{-st}}{1+s^2}.
\end{aligned}
$$

Let $H(t) = -\sin(t)\frac{se^{-st}}{1+s^2} - \cos(t)\frac{e^{-st}}{1+s^2}$. From (4), we have $H'(t) = \sin(t)e^{-st}$ and we get by integrating by parts

$$\begin{aligned}
\int_0^\infty t\sin(t)e^{-st}dt &= \int_0^\infty tH'(t)dt \\
&= [tH(t)]_0^\infty - \int_0^\infty H(t)dt = -\int_0^\infty H(t)dt \\
&= \int_0^\infty \Big(\sin(t)\frac{se^{-st}}{1+s^2} + \cos(t)\frac{e^{-st}}{1+s^2}\Big)dt \\
&= \frac{s}{1+s^2}\int_0^\infty \sin(t)e^{-st}dt + \frac{1}{1+s^2}\int_0^\infty \cos(t)e^{-st}dt \\
&= \frac{s}{1+s^2}\mathcal{L}(\sin(t)) + \frac{1}{1+s^2}\mathcal{L}(\cos(t)) \\
&= \frac{s}{1+s^2}\frac{1}{1+s^2} + \frac{1}{1+s^2}\frac{s}{1+s^2} = \frac{2s}{(1+s^2)^2}. \qquad (5)
\end{aligned}$$

From (1) and (5), we get $\mathcal{L}(f(t)) = \dfrac{2s}{(1+s^2)^2}$ □

25. Let $f(t) = (t+1)^3 = t^3 + 3t^2 + 3t + 1$. Then we have

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(t^3) + 3\mathcal{L}(t^2) + 3\mathcal{L}(t) + \mathcal{L}(1) \\
&= \frac{3!}{s^4} + 3\frac{2!}{s^3} + 3\frac{1!}{s^2} + \frac{1}{s} \\
&= \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}.
\end{aligned}$$

□

30. Let $f(t) = (e^t - e^{-t})^2 = e^{2t} + e^{-2t} - 2$. Then we have

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(e^{2t}) + \mathcal{L}(e^{-2t}) - 2\mathcal{L}(1) \\
&= \frac{1}{s-2} + \frac{1}{s+2} - \frac{2}{s} \\
&= \frac{8}{s(s^2-4)}.
\end{aligned}$$

□

38. Let $f(t) = \cos^2(t) = \frac{1}{2}(1 + \cos(2t))$. Then we have

$$\begin{aligned}\mathcal{L}(f(t)) &= \frac{1}{2}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(\cos(2t)) \\ &= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{s^2+4} \\ &= \frac{s^2+2}{s(s^2+4)}.\end{aligned}$$

□

40. Let $f(t) = 10\cos\left(t - \frac{\pi}{6}\right)$. Note that

$$\begin{aligned}f(t) &= 10\Big(\cos(t)\cos\left(\frac{\pi}{6}\right) + \sin(t)\sin\left(\frac{\pi}{6}\right)\Big) \\ &= 10\Big(\frac{\sqrt{3}}{2}\cos(t) + \frac{1}{2}\sin(t)\Big) \\ &= 5\sqrt{3}\cos(t) + 5\sin(t).\end{aligned}$$

Then we have

$$\begin{aligned}\mathcal{L}(f(t)) &= 5\sqrt{3}\mathcal{L}(\cos(t)) + 5\mathcal{L}(\sin(t)) \\ &= 5\sqrt{3}\frac{s}{s^2+1} + 5\frac{1}{s^2+1} \\ &= \frac{5(\sqrt{3}s+1)}{(s^2+1)}.\end{aligned}$$

□