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MATH 531/Term 062/Hw#1(Chap. 1)/

6. Let $f : X \to Y$ be a mapping of a nonempty set X into a nonempty set Y. We would like to show that f is one-to-one if and only if there is a mapping $g : Y \to X$ such that $gof = Id_X$.

($\Rightarrow$) Assume that f is one-to-one. Since $X \neq \emptyset$, there exists $x_0 \in X$. Moreover since f is one-to-one, for each $y \in f(X)$ the set $f^{-1}(\{y\})$ contains a unique element x. We define a function $g : Y \to X$ in the following way

$$\begin{cases} g(y) = x, & \text{if } \ y \in f(X), \text{ where } x \text{ is the unique element of } \ f^{-1}(\{y\}) \\ g(y) = x_0, & \text{if } \ y \notin f(X), \end{cases}$$

Then we have for any $x \in X$, $gof(x) = g(f(x)) = x$. Hence $gof = Id_X$.

($\Leftarrow$) Assume that there exists a mapping $g : Y \to X$ such that $gof = Id_X$. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then we have $g(f(x_1)) = g(f(x_2))$ which leads to $gof(x_1) = gof(x_2)$ or $Id_X(x_1) = Id_X(x_2)$. Thus $x_1 = x_2$ and f is one-to-one.

□

7. Let $f : X \to Y$ be a mapping of a nonempty set X into a nonempty set Y. We would like to show that f is onto if and only if there is a mapping $g : Y \to X$ such that $fog = Id_Y$.

($\Rightarrow$) Assume that f is onto. Then for each $y \in Y$, the set $f^{-1}(\{y\})$ is not empty. Consider then the collection of sets $\{f^{-1}(\{y\})\}_{y \in Y}$. By the axiom of choice there exists a function G which assigns to each set $f^{-1}(\{y\})$ an element $G(y) \in f^{-1}(\{y\})$. We define then a function $g : Y \to X$ by setting $g(y) = G(y)$ for each $y \in Y$. Moreover g satisfies for each $y \in Y$, $fog(y) = f(g(y)) = y$ since $g(y) \in f^{-1}(\{y\})$. Hence g satisfies $fog = Id_Y$.

($\Leftarrow$) Assume that there exists a mapping $g : Y \to X$ such that $fog = Id_Y$. Let $y \in Y$ and set $x = g(y)$. Then we have $f(x) = f(g(y)) = fog(y) = Id_Y(y) = y$. Hence f is onto.

□

13. Let $(A_i)_{i\in I}$ be a collection of subsets of a set E. We would like to prove the following De Morgan's laws:

$$\Big[\bigcup_{i\in I} A_i\Big]^c = \bigcap_{i\in I} A_i^c \tag{1}$$

$$\Big[\bigcap_{i\in I} A_i\Big]^c = \bigcup_{i\in I} A_i^c. \tag{2}$$

For each $x \in E$, we have

$$\begin{aligned} x \in \Big[\bigcup_{i\in I} A_i\Big]^c \quad &\Leftrightarrow \quad x \notin \bigcup_{i\in I} A_i \;\Leftrightarrow\; \forall i \in I \; x \notin A_i \\ &\Leftrightarrow \quad \forall i \in I \; x \in A_i^c \;\Leftrightarrow\; x \in \bigcap_{i\in I} A_i^c. \end{aligned}$$

This proves (1).

For each $x \in E$, we have

$$\begin{aligned} x \in \Big[\bigcap_{i\in I} A_i\Big]^c \quad &\Leftrightarrow \quad x \notin \bigcap_{i\in I} A_i \;\Leftrightarrow\; \exists i \in I \; x \notin A_i \\ &\Leftrightarrow \quad \exists i \in I \; x \in A_i^c \;\Leftrightarrow\; x \in \bigcup_{i\in I} A_i^c. \end{aligned}$$

This proves (2).

□

16. Let $f : E \to F$ be a mapping of a nonempty set E into a nonempty set F. Let $(A_i)_{i\in I}$ be a collection of subsets of E. We would like to prove the following:

$$f\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f(A_i) \tag{1}$$

$$f\Big(\bigcap_{i\in I} A_i\Big) \subset \bigcap_{i\in I} f(A_i). \tag{2}$$

a) For each $y \in F$, we have

$$y \in f\Big(\bigcup_{i\in I} A_i\Big) \;\Leftrightarrow\; \exists x \in \bigcup_{i\in I} A_i \;:\, f(x) = y \;\Leftrightarrow\; \exists i \in I\ \exists x \in A_i \;:\, f(x) = y$$
$$\Leftrightarrow\; \exists i \in I \;:\; y \in f(A_i) \;\Leftrightarrow\; y \in \bigcup_{i\in I} f(A_i).$$

This proves (1).

b) For each $y \in F$, we have

$$\begin{aligned} y \in f\Big(\bigcap_{i\in I} A_i\Big) &\Rightarrow \exists x \in \bigcap_{i\in I} A_i \;:\, f(x) = y \\ &\Rightarrow \exists x \in E \;:\; \forall i \in I\ x \in A_i \text{ and } f(x) = y \\ &\Rightarrow \forall i \in I \;:\; y \in f(A_i) \\ &\Rightarrow y \in \bigcap_{i\in I} f(A_i). \end{aligned}$$

This proves (2).

c) Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 1$ for all $n \geq 1$. Then we have for $A_n = \{n\}$, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ and $\bigcap_{n\in\mathbb{N}} f(A_n) = \{1\}$. Therefore

$$f(\bigcap_{n\in\mathbb{N}} A_n) = f(\emptyset) = \emptyset \neq \bigcap_{n\in\mathbb{N}} f(A_n) = \{1\}.$$

□

17. Let $f : E \to F$ be a mapping of a nonempty set E into a nonempty set F. Let $(B_i)_{i\in I}$ be a collection of subsets of F. We would like to prove the following:

$$f^{-1}\Big(\bigcup_{i\in I} B_i\Big) = \bigcup_{i\in I} f^{-1}(B_i) \tag{1}$$

$$f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i) \tag{2}$$

$$\forall B \subset F\ f^{-1}(B^c) = \Big[f^{-1}(B)\Big]^c. \tag{3}$$

a) For each $x \in E$, we have

$$\begin{aligned} x \in f^{-1}\Big(\bigcup_{i\in I} B_i\Big) &\Leftrightarrow f(x) \in \bigcup_{i\in I} B_i \Leftrightarrow \exists i \in I \ : f(x) \in B_i \\ &\Leftrightarrow \exists i \in I \ x \in f^{-1}(B_i) \\ &\Leftrightarrow x \in \bigcup_{i\in I} f^{-1}(B_i). \end{aligned}$$

This proves (1).

b) For each $x \in E$, we have

$$\begin{aligned} x \in f^{-1}\Big(\bigcap_{i\in I} B_i\Big) &\Leftrightarrow f(x) \in \bigcap_{i\in I} B_i \Leftrightarrow \forall i \in I \ : f(x) \in B_i \\ &\Leftrightarrow \forall i \in I \ x \in f^{-1}(B_i) \\ &\Leftrightarrow x \in \bigcap_{i\in I} f^{-1}(B_i). \end{aligned}$$

This proves (2).

c) Let $B \subset F$. Then we have for each $x \in E$

$$\begin{aligned} x \in f^{-1}(B^c) &\Leftrightarrow f(x) \in B^c \Leftrightarrow f(x) \notin B \\ &\Leftrightarrow x \notin f^{-1}(B) \\ &\Leftrightarrow x \in \Big[f^{-1}(B)\Big]^c. \end{aligned}$$

□

21. Let $f : X \to Y$ be a mapping of a nonempty set X onto a nonempty set Y. We would like to show that there exists a mapping $g : Y \to X$ such that $fog = Id_Y$.

Since f is onto, for each $y \in Y$, the set $f^{-1}(\{y\})$ is not empty. Consider then the collection of sets $\{f^{-1}(\{y\})\}_{y\in Y}$. By the axiom of choice there exists a function G which assigns to each set $f^{-1}(\{y\})$ an element $G(y) \in f^{-1}(\{y\})$. We define then a function $g : Y \to X$ by setting $g(y) = G(y)$ for each $y \in Y$. Moreover g satisfies for each $y \in Y$, $fog(y) = f(g(y)) = y$ since $g(y) \in f^{-1}(\{y\})$. Hence g satisfies $fog = Id_Y$.

26. Let X be an infinite set. We would like to show that there exists a countably infinite subset of X. To do this we consider a choice function $F : \mathcal{P}(X) \to X$ such that $F(A) \in A$ for any $A \in \mathcal{P}(X)$. Then we define a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X by the following

$$\begin{cases} x_1 = F(X) \in X, \\ x_{n+1} = F(X \setminus \{x_1, ..., x_n\}) \in X \setminus \{x_1, ..., x_n\}, \quad \forall n \geq 1. \end{cases}$$

The sequence is well defined because for each $n \in \mathbb{N}$, the set $X \setminus \{x_1, ..., x_n\}$ is not empty. Indeed if $X \setminus \{x_1, ..., x_n\} = \emptyset$, then $X = \{x_1, ..., x_n\}$ is finite which contradicts the fact that X is an infinite set.

Now we set $E = \{x_n \ / \ n \in \mathbb{N} \ \}$ and we consider the mapping $f : \mathbb{N} \to E$ defined by $f(n) = x_n$. Then f is a bijection:

f is injective: Let $n, m \in \mathbb{N}$ such that $n \neq m$. Assume for example that $n < m$ or $n \leq m - 1$. By construction we have $x_m \in X \setminus \{x_1, ..., x_{m-1}\}$. Since $x_n \in \{x_1, ..., x_{m-1}\}$, we have necessarily $x_n \neq x_m$ i.e. $f(n) \neq f(m)$.

f is surjective: Let $x \in E$. By definition of E there exists $n \in \mathbb{N}$ such that $x = x_n$ i.e. there exists $n \in \mathbb{N}$ such that $x = f(n)$. $\square$

Hence E is a countably infinite subset of X.