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MATH 301/Term 062/Hw#1(9.1)/

12. Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be the vector function that describes the curve C of intersection of the surfaces:

$$x^2 + y^2 - z^2 = 1, \quad y = 2x, \quad \text{and} \quad x = t.$$

It is clear by substitution that

$$\begin{cases} f(t) = t, \\ g(t) = 2t, \\ h^2(t) = f^2(t) + g^2(t) - 1 = 5t^2 - 1. \end{cases}$$

Hence C is given by

$$\begin{cases} f(t) = t, \\ g(t) = 2t, \\ h(t) = \sqrt{5t^2 - 1}, \end{cases} \quad t \in [\tfrac{\sqrt{5}}{5}, \infty) \qquad \text{or} \qquad \begin{cases} f(t) = t, \\ g(t) = 2t, \\ h(t) = -\sqrt{5t^2 - 1}. \end{cases} \quad t \in [\tfrac{\sqrt{5}}{5}, \infty)$$

□

20. Let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + \tan^{-1}(t)\mathbf{k}$. It is clear that $\mathbf{r}(t)$ is defined and twice differentiable everywhere with:

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$
$$\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} - \frac{2t}{(1+t^2)^2}\mathbf{k}.$$

□

26. Let C be the curve given by

$$x = t^3 - t, \quad y = \frac{6t}{t+1}, \quad \text{and} \quad z = (2t+1)^2.$$

The vector function that represents C is given by

$$\mathbf{r}(t) = (t^3 - t)\mathbf{i} + \frac{6t}{t+1}\mathbf{j} + (2t+1)^2\mathbf{k}.$$

$\mathbf{r}(t)$ is defined and differentiable except at $t = -1$ with:

$$\mathbf{r}'(t) = (3t^2 - 1)\mathbf{i} + \frac{6}{(t+1)^2}\mathbf{j} + 4(2t+1)\mathbf{k}$$

and

$$\mathbf{r}'(1) = 2\mathbf{i} + \frac{3}{2}\mathbf{j} + 12\mathbf{k}$$

is tangent to C at $P = \mathbf{r}(1) = 3\mathbf{j} + 9\mathbf{k}$.
Now let $M = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be an arbitrary point of the tangent line to C at the point P. We know that the vectors $\overrightarrow{PM}$ and $\mathbf{r}'(1)$ are linearly dependent. Therefore there exists t such that

$$\begin{cases} x - 0 = 2t, \\ y - 3 = \frac{3}{2}t, \\ z - 9 = 12t. \end{cases} \qquad \text{or} \qquad \begin{cases} x = 2t, \\ y = \frac{3}{2}t + 3, \\ z = 12t + 9. \end{cases} \quad t \text{ is an arbitrary real number.}$$

$\square$

28. Let $\mathbf{r}(t)$ be a differentiable vector function on some open interval. Then we have

$$\begin{aligned} \frac{d}{dt}\big[\mathbf{r}(t).(t\mathbf{r}(t))\big] &= \big[\frac{d}{dt}\mathbf{r}(t)\big].(t\mathbf{r}(t)) + \mathbf{r}(t).\frac{d}{dt}\big[(t\mathbf{r}(t))\big] \\ &= \mathbf{r}'(t).(t\mathbf{r}(t)) + \mathbf{r}(t).\big[t\mathbf{r}'(t) + \mathbf{r}(t)\big] \\ &= \mathbf{r}'(t).(t\mathbf{r}(t)) + \mathbf{r}(t).(t\mathbf{r}'(t)) + \mathbf{r}(t).\mathbf{r}(t) \\ &= 2t\mathbf{r}(t).\mathbf{r}'(t) + \mathbf{r}(t).\mathbf{r}(t). \end{aligned}$$

$\square$

41. Let $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + ct\mathbf{k},\ t \in [0, 2\pi]$. It is clear that $\mathbf{r}(t)$ is defined and differentiable everywhere with:

$$\mathbf{r}'(t) = -a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j} + c\mathbf{k}.$$

The length of the curve traced by $\mathbf{r}(t)$ is given by:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + c^2}dt \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t) + c^2}dt \\ &= \int_0^{2\pi} \sqrt{a^2 + c^2}dt = 2\pi\sqrt{a^2 + c^2}. \end{aligned}$$

□