King Fahd University of Petroleum and Minerals

Department of Mathematical Sciences

MATH 301/Exam 1/ Term 032/ Time allowed=2 Hours

1. 1. Find the total length of the curve traced by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \cos t\mathbf{i} + \sin t\mathbf{j} + \frac{2}{3}t^{\frac{3}{2}}\mathbf{k}$, $0 \le t \le 3$.

Solution:

The total length of the curve is given by

$$\begin{aligned} L &= \int_0^3 \sqrt{f'^2(t) + g'^2(t) + h'^2(t)}dt \\ &= \int_0^3 \sqrt{\sin^2(t) + \cos^2(t) + (t^{1/2})^2}dt \\ &= \int_0^3 \sqrt{1+t}dt = \Big[\frac{2}{3}(1+t)^{3/2}\Big]_0^3 = \frac{2}{3}\big[4^{3/2} - 1\big] = \frac{14}{3}. \end{aligned}$$

□

2. Show that $(x^2 + \ln(y^2+z^2))dx + \Big(y^2 + \frac{2xy}{y^2+z^2}\Big)dy + \Big(z^2 + \frac{2xz}{y^2+z^2}\Big)dz$ is an exact differential.

Calculate $\displaystyle\int_{(0,0,1)}^{(1,1,1)} (x^2 + \ln(y^2+z^2))dx + \Big(y^2 + \frac{2xy}{y^2+z^2}\Big)dy + \Big(z^2 + \frac{2xz}{y^2+z^2}\Big)dz.$

Solution:

a) Let $P(x,y,z) = x^2 + \ln(y^2+z^2$, $Q(x,y,z) = y^2 + \frac{2xy}{y^2+z^2}$ and $R(x,y,z) = z^2 + \frac{2xz}{y^2+z^2}$. These functions are continuous and have partial derivatives continuous on any domain which does not contain the x-axis. Moreover we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{2y}{y^2+z^2}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{2z}{y^2+z^2} \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = \frac{-4xyz}{(y^2+z^2)^2}.$$

Hence $(x^2+\ln(y^2+z^2))dx+\Big(y^2+\dfrac{2xy}{y^2+z^2}\Big)dy+\Big(z^2+\dfrac{2xz}{y^2+z^2}\Big)dz$ is an exact differential.

b) It follows from a) that there exists a function ϕ such that

$$d\phi=(x^2+\ln(y^2+z^2))dx+\Big(y^2+\frac{2xy}{y^2+z^2}\Big)dy+\Big(z^2+\frac{2xz}{y^2+z^2}\Big)dz.$$

Then we have

$$\frac{\partial\phi}{\partial x}=x^2+\ln(y^2+z^2) \tag{1}$$
$$\frac{\partial\phi}{\partial y}=y^2+\frac{2xy}{y^2+z^2} \tag{2}$$
$$\frac{\partial\phi}{\partial z}=z^2+\frac{2xz}{y^2+z^2}. \tag{3}$$

Integrating (1), we get

$$\phi(x,y,z)=\int(x^2+\ln(y^2+z^2))dx=\frac{1}{3}x^3+x\ln(y^2+z^2)+g(y,z). \tag{4}$$

Using (2) and (4), we get

$$\frac{2xy}{y^2+z^2}+\frac{\partial g}{\partial y}=y^2+\frac{2xy}{y^2+z^2}\ \Rightarrow\ \frac{\partial g}{\partial y}=y^2\ \Rightarrow\ g(y,z)=\frac{1}{3}y^3+h(z)$$
$$\Rightarrow\ \phi(x,y,z)=\frac{1}{3}x^3+\frac{1}{3}y^3+x\ln(y^2+z^2)+h(z). \tag{5}$$

Using (3) and (5), we get

$$\frac{2xz}{y^2+z^2}+h'(z)=z^2+\frac{2xz}{y^2+z^2}\ \Rightarrow\ h'(z)=z^2\ \Rightarrow\ h(z)=\frac{1}{3}z^3+C. \tag{6}$$

Combining (5) and (6), we obtain

$$\phi(x,y,z)=\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{3}z^3+x\ln(y^2+z^2)+C.$$

Since the line integral is independent of the path, we obtain

$$\int_{(0,0,1)}^{(1,1,1)} (x^2+\ln(y^2+z^2))dx + \Big(y^2+\frac{2xy}{y^2+z^2}\Big)dy + \Big(z^2+\frac{2xz}{y^2+z^2}\Big)dz$$
$$= \phi(1,1,1)-\phi(0,0,1) = \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\ln(2)-\frac{1}{3}-\ln(1) = \frac{2}{3}+\ln(2).$$

□

3. Let C be the triangle with vertices $(0,0)$, $(2,0)$ and $(2,1)$. Verify the Green theorem for $2e^y dx + xe^y dy$.

Solution:

Let $P(x,y) = 2e^y$, $Q(x,y) = xe^y$ and R be the region of the $xy-$plane bounded by the triangle C with vertices $(0,0)$, $(2,0)$ and $(2,1)$ (draw a figure). We would like to verify the following Green's formula

$$\oint_C Pdx + Qdy = \int\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy. \tag{1}$$

First note that $P(x,y)$ and $Q(x,y)$ are continuous and have partial derivatives continuous on any domain. Moreover we have $\frac{\partial P}{\partial y} = 2e^y$ and $\frac{\partial Q}{\partial x} = e^y$ Next we have for the right-hand side of (1)

$$\begin{aligned}\int\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy &= \int\int_R -e^y dxdy = -\int_0^2 \Big(\int_0^{x/2} e^y dy\Big)dx \\ &= -\int_0^2 \big[e^y\big]_0^{x/2} dx = -\int_0^2 (e^{x/2}-1)dx \\ &= -\big[2e^{x/2}-x\big]_0^2 = -(2e-2-2) = 4-2e. \end{aligned} \tag{2}$$

Now we evaluate the left-hand side of (1). Note that $C = C_1 \cup C_2 \cup C_3$, where C_1 is the horizontal line segment joining the points $(0,0)$ and $(2,0)$, C_2 is the vertical line segment joining the points $(2,0)$ and $(2,1)$, and where C_3 is line segment joining the points $(2,1)$ and $(0,0)$. C_1, C_2 and C_3 have the parameterizations

$$C_1 : \begin{cases} x = t, \\ y = 0, \quad t \in [0,2], \end{cases} \quad C_2 : \begin{cases} x = 2, \\ y = t, \quad t \in [0,1], \end{cases} \quad C_3 : \begin{cases} x = t, \\ y = t/2, \quad t \in [0,2]. \end{cases}$$

Then we have

$$\oint_C 2e^y dx + xe^y dy = \int_{C_1} 2e^y dx + xe^y dy + \int_{C_2} 2e^y dx + xe^y dy + \int_{C_3} 2e^y dx + xe^y dy. \tag{3}$$

$$\begin{aligned} \int_{C_1} 2e^y dx + xe^y dy &= \int_{C_1} 2e^y dx + \int_{C_1} xe^y dy \\ &= \int_0^2 2e^0 dt + \int_0^2 te^0(0) dt \\ &= [2t]_0^2 = 4. \end{aligned} \tag{4}$$

$$\begin{aligned} \int_{C_2} 2e^y dx + xe^y dy &= \int_{C_2} 2e^y dx + \int_{C_2} xe^y dy \\ &= \int_0^1 2e^t(0) dt + \int_0^1 2e^t dt \\ &= [2e^t]_0^1 = 2e - 2. \end{aligned} \tag{5}$$

$$\begin{aligned} \int_{C_3} 2e^y dx + xe^y dy &= \int_{C_3} 2e^y dx + \int_{C_3} xe^y dy \\ &= -\int_0^2 2e^{t/2} dt - \int_0^2 te^{t/2}(1/2) dt \\ &= -[4e^{t/2}]_0^2 - [te^{t/2} - 2e^{t/2}]_0^2 \\ &= -[4e - 4] - [2e - 2e + 2] = -4e + 2. \end{aligned} \tag{6}$$

Using (3), (4), (5) and (6), we get

$$\int_C 2e^y dx + xe^y dy = 4 + 2e - 2 - 4e + 2 = 4 - 2e. \tag{7}$$

Comparing (2) and (7), we conclude that (1) is satisfied.

□

4. Evaluate the area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ that is within the cylinders $x^2 + y^2 = b^2$ and $x^2 + y^2 = c^2$, with $0 < b < c < a$.

Solution:

Let S be the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that is within the cylinders $x^2 + y^2 = b^2$ and $x^2 + y^2 = c^2$ (draw a figure). We have $S = S^+ \cup S^-$, where S^+ (resp. S^-) is the part of S located above (resp. below) the $xy-$plane. By symmetry we have $Area(S^+) = Area(S^-)$ and therefore $Area(S) = 2Area(S^+)$.
Since S^+ is defined by the equation $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$, the area of S^+ is given by

$$Area(S^+) = \int\int_R \sqrt{1 + f_x^2 + f_y^2}dxdy, \qquad (1)$$

where R is the projection of S^+ on the $xy-$plane. Moreover we have

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

We deduce then from (1) by using the polar coordinates

$$\begin{aligned} Area(S^+) &= \int\int_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}dxdy \\ &= a\int_0^{2\pi}\int_b^c \frac{rdrd\theta}{\sqrt{a^2 - r^2}} \\ &= a\int_0^{2\pi} \big[-\sqrt{a^2 - r^2}\big]_b^c d\theta \\ &= 2\pi a(\sqrt{a^2 - b^2} - \sqrt{a^2 - c^2}). \end{aligned}$$

Hence $Area(S) = 2Area(S^+) = 4\pi a(\sqrt{a^2 - b^2} - \sqrt{a^2 - c^2})$.

□

5. Evaluate the line integral $\oint_C y^2dx + z^2dy + x^2dz$, where C is the trace of the cylinder $y^2 + z^2 = 1$ on the plane $x + z = 6$.

Solution:

Let $\mathbf{F} = y2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ and let S be the portion of the plane $x + z = 6$ that is located inside the cylinder $y^2 + z^2 = 1$ (draw a figure). S is defined by the equation $z = f(x, y) = 6 - x$. The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. Therefore we have by Stokes' theorem

$$\oint_C y^2dx + z^2dy + x^2dz = \oint_C \mathbf{F}.d\mathbf{r} = \int\int_S curl(\mathbf{F}).\mathbf{n}dS, \tag{1}$$

where

$$\begin{aligned} curl\mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y2 & z^2 & x^2 \end{vmatrix} \\ &= (0 - 2z)\mathbf{i} - (2x - 0)\mathbf{j} + (0 - 2y)\mathbf{k} \\ &= -2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}. \end{aligned} \tag{2}$$

The unit normal vector to S is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\big(- f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}\big) = \frac{1}{\sqrt{2}}\big(\mathbf{i} + \mathbf{k}\big). \tag{3}$$

Now let R be the projection of S on the $xz-$plane. Then we get by using (1), (2) and (3)

$$\begin{aligned} \oint_C y^2dx + z^2dy + x^2dz &= \int\int_S -\frac{2}{\sqrt{2}}(y + z)dS \\ &= \int\int_R -\sqrt{2}(y + z)\sqrt{2}dydz \quad \text{since } S \text{ is also defined by } x = 6 - z \\ &= -2\int\int_R (y + z)dydz. \end{aligned} \tag{4}$$

Using the polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we obtain

$$
\begin{aligned}
\int\int_R (y+z)dydz &= \int_0^{2\pi}\Big(\int_0^1 (r\cos\theta + r\sin\theta) r dr\Big) d\theta \\
&= \int_0^{2\pi}\Big(\int_0^1 \Big[\frac{1}{2}r^2\cos\theta + \frac{1}{2}r^2\sin\theta\Big]_0^1\Big) d\theta \\
&= \int_0^{2\pi}\Big(\frac{1}{2}\cos\theta + \frac{1}{2}\sin\theta\Big) d\theta \\
&= \Big[\frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta\Big]_0^{2\pi} = 0. \qquad (5)
\end{aligned}
$$

Hence we deduce from (1), (4) and (5) that

$$\oint_C yz^2dx + xz^2dy + x^2ydz = 0.$$

□

6. Let S be the exterior boundary of the region D that is above the $xy-$plane, and bounded by the cylinder $x^2+z^2=1$ and the planes $y=1$, $y=5$. Verify the divergence theorem for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution:

The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. We would like to verify the divergence formula i.e.

$$\int\int_S \mathbf{F}.\mathbf{n}\ dS = \int\int\int_D div(\mathbf{F})dV. \qquad (1)$$

First we have for the right-hand side of (1)

$$\int\int\int_D div(\mathbf{F})dV = \int\int\int_D 3dV = 3Vol(D) = 3\frac{1}{2}(\pi 1^2)(5-1) = 6\pi. \qquad (2)$$

Next we have for the left-hand side of (1)

$$\begin{aligned}\int\int_S \mathbf{F}.\mathbf{n}\ dS &= \int\int_{S_1} \mathbf{F}.\mathbf{n}\ dS + \int\int_{S_2} \mathbf{F}.\mathbf{n}\ dS \\ &+ \int\int_{S_3} \mathbf{F}.\mathbf{n}\ dS + \int\int_{S_4} \mathbf{F}.\mathbf{n}\ dS \end{aligned} \tag{3}$$

S_1 is defined by $y = 1$ and the unit normal vector to S_1 is given by $\mathbf{n} = -\mathbf{j}$. So we have

$$\int\int_{S_1} \mathbf{F}.\mathbf{n}\ dS = \int\int_{S_1} -y\ dS = -\int\int_{S_1} dS = -Area(S_1) = -\pi/2. \tag{4}$$

S_2 is defined by $y = 5$ and the unit normal vector to S_2 is given by $\mathbf{n} = \mathbf{j}$. So we have

$$\int\int_{S_2} \mathbf{F}.\mathbf{n}\ dS = \int\int_{S_2} y\ dS = 5\int\int_{S_2} dS = 5Area(S_2) = 5\pi/2. \tag{5}$$

S_3 is defined by $z = 0$ and the unit normal vector to S_3 is given by $\mathbf{n} = -\mathbf{k}$. So we have

$$\int\int_{S_3} \mathbf{F}.\mathbf{n}\ dS = \int\int_{S_3} -z\ dS = \int\int_{S_3} 0dS = 0. \tag{6}$$

S_4 is defined by $g(x, y, z) = 0$, where $g(x, y, z) = x^2 + z^2 - 1$. So the unit normal vector to S_4 is given by

$$\mathbf{n} = \frac{1}{||\nabla g||}\nabla g = \frac{1}{\sqrt{4x^2+4z^2}}(2x\mathbf{i}+2z\mathbf{k}) = \frac{1}{\sqrt{x^2+z^2}}(x\mathbf{i}+z\mathbf{k}) = x\mathbf{i}+z\mathbf{k}. \tag{7}$$

Then we have

$$\int\int_{S_4} \mathbf{F}.\mathbf{n}\ dS = \int\int_{S_4} dS = Area(S_4) = \frac{1}{2}2\pi(1)(5-1) = 4\pi. \tag{8}$$

Taking into account (3)-(8), we get

$$\int\int_S \mathbf{F}.\mathbf{n}\ dS = -\pi/2 + 5\pi/2 + 0 + 4\pi = 6\pi. \tag{9}$$

Comparing (2) and (9), we conclude that (1) is satisfied.

□