King Fahd University of Petroleum and Minerals

Department of Mathematical Sciences

MATH 301/Exam 1/ Term 031/ Time allowed=2 Hours

1. Find the directional derivative of $f(x, y) = e^{-xy} \cos(x)$ at $(\pi, 1)$ in the direction $u = -\mathbf{i} + \mathbf{j}$.

<u>**Solution:**</u>

We would like to find $D_u f(x, y)$. Since u is not a unit vector we have

$$D_u f(x, y) = \nabla f(x, y).\frac{1}{||u||}u.$$

Now

$$\begin{aligned}
\nabla f(x, y) &= -e^{-xy}(y\cos(x) + \sin(x))\mathbf{i} - xe^{-xy}\cos(x)\mathbf{j} \\
\nabla f(\pi, 1) &= -e^{-\pi}(\cos(\pi) + \sin(\pi))\mathbf{i} - \pi e^{-\pi}\cos(\pi)\mathbf{j} = e^{-\pi}\mathbf{i} + \pi e^{-\pi}\mathbf{j} \\
\frac{1}{||u||}u &= \frac{1}{\sqrt{1+1}}(-\mathbf{i} + \mathbf{j}) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}.
\end{aligned}$$

Hence

$$D_u f(\pi, 1) = e^{-\pi}\frac{\sqrt{2}}{2}(\mathbf{i} + \pi\mathbf{j}).(-\mathbf{i} + \mathbf{j}) = (\pi - 1)e^{-\pi}\frac{\sqrt{2}}{2}.$$

□

2. Find an equation of the tangent plane to the graph of $z = \frac{1}{3}x^3 + \frac{1}{3}y^3 + 1$ at $(1, 2, -1)$.

<u>**Solution:**</u>

The equation of the tangent plane to the graph of $z = f(x, y)$ at (x_0, y_0, z_0) is given by

$$-(x - x_0)f_x(x_0, y_0) - (y - y_0)f_y(x_0, y_0) + (z - z_0) = 0. \tag{1}$$

Here $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + 1$ and $(x_0, y_0, z_0) = (1, 2, -1)$.
We have $f_x(x,y) = x^2$ and $f_y(x,y) = y^2$. In particular we have $f_x(1,2) = 1$ and $f_y(1,2) = 4$. Therefore we deduce from (1) the following equation of the tangent plane at $(1, 2, -1)$

$$-(x-1) - 4(y-2) + (z+1) = 0 \quad \Leftrightarrow \quad x + 4y - z = 10.$$

□

3. Show that the line integral $\oint_C (y^2 + z^2)dx + 2xydy + 2xzdz$ is independent of the path.

Solution:

The line integral $\oint_C (y^2 + z^2)dx + 2xydy + 2xzdz$ is of the form $\int_C Pdx + Qdy + Rdz$, with $P(x,y,z) = y^2 + z^2$, $Q(x,y,z) = 2xy$ and $R(x,y,z) = 2xz$ which are continuous and have partial derivatives continuous on any domain. Moreover we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2y, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 2z \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 0.$$

Hence the line integral is independent of the path.

□

4. Find ϕ such that $d\phi = (y^2 + z^2)dx + 2xydy + 2xzdz$.

Solution:

Let ϕ be a function such that $d\phi = (y^2 + z^2)dx + 2xydy + 2xzdz$. Then we have

$$\frac{\partial \phi}{\partial x} = y^2 + z^2 \qquad (1)$$

$$\frac{\partial \phi}{\partial y} = 2xy \qquad (2)$$

$$\frac{\partial \phi}{\partial z} = 2xz. \qquad (3)$$

Integrating (1), we get

$$\phi(x,y,z) = \int (y^2+z^2)dx = x(y^2+z^2) + g(y,z). \tag{4}$$

Using (2) and (4), we get

$$2xy + \frac{\partial g}{\partial y} = 2xy \;\Rightarrow\; \frac{\partial g}{\partial y} = 0 \;\Rightarrow\; g(y,z) = h(z) \;\Rightarrow\; \phi(x,y,z) = x(y^2+z^2) + h(z). \tag{5}$$

Using (3) and (5), we get

$$2xz + h'(z) = 2xz \;\Rightarrow\; h'(z) = 0 \;\Rightarrow\; h(z) = C. \tag{6}$$

Combining (5) and (6), we obtain

$$\phi(x,y,z) = x(y^2+z^2) + C.$$

□

5. Evaluate the line integral $\oint_C -ydx + xdy$, where C is the Cardioid defined by $x = \cos\theta(1+\cos\theta)$, $y = \sin\theta(1+\cos\theta)$, $\theta \in [0, 2\pi]$. Then deduce the area of the region bounded by C.

Solution:

The curve C is defined by

$$C : \begin{cases} x = \cos\theta(1+\cos\theta), \\ y = \sin\theta(1+\cos\theta), \quad \theta \in [0, 2\pi]. \end{cases}$$

We have

$$\int_C -ydx + xdy = \int_C -ydx + \int_C xdy. \tag{1}$$

$$\begin{aligned} \int_C -ydx &= \int_0^{2\pi} -\sin\theta(1+\cos\theta)(-\sin\theta - 2\sin\theta\cos\theta)d\theta \\ &= \int_0^{2\pi} (\sin^2\theta + 3sin^2\theta\cos\theta + 2\sin^2\theta\cos^2\theta). \qquad (2) \end{aligned}$$

Note that

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \tag{3}$$

$$\sin^2\theta\cos^2\theta) = \frac{1}{4}\sin^2(2\theta) = \frac{1}{8}(1-\cos(4\theta)). \tag{4}$$

Using (3) and (4), we get from (2)

$$\begin{aligned}
\int_C -ydx &= \int_0^{2\pi}\Big(\frac{1}{2}(1-\cos(2\theta)) + 3sin^2\theta\cos\theta + \frac{1}{4}(1-\cos(4\theta))\Big) \\
&= \int_0^{2\pi}\Big(\frac{3}{4} + 3sin^2\theta\cos\theta - \frac{1}{2}\cos(2\theta) - \frac{1}{4}\cos(4\theta)\Big) \\
&= \Big[\frac{3}{4}\theta + sin^3\theta - \frac{1}{4}\sin(2\theta) - \frac{1}{16}\sin(4\theta)\Big]_0^{2\pi} \\
&= \frac{3}{2}\pi.
\end{aligned} \tag{5}$$

Now we have for the second integral in the right hand-side of (1)

$$\begin{aligned}
\int_C xdy &= \int_0^{2\pi}\cos\theta(1+\cos\theta)(\cos\theta + \cos^2\theta - \sin^2\theta)d\theta \\
&= \int_0^{2\pi}(\cos^2\theta + 2\cos^3\theta + \cos^4\theta - sin^2\theta\cos\theta - \sin^2\theta\cos^2\theta).
\end{aligned} \tag{6}$$

Note that

$$\cos^2\theta = \frac{1}{2}(\cos(2\theta)+1) \tag{7}$$

$$\cos^3\theta = \cos\theta(1-\sin^2\theta) = \cos\theta - \sin^2\theta\cos\theta \tag{8}$$

$$\cos^4\theta = \cos\theta\cos^3\theta = \cos^2\theta - \sin^2\theta\cos^2\theta. \tag{9}$$

Using (4), (7), (8) and (9), we get from (6)

$$\begin{aligned}
\int_C xdy &= \int_0^{2\pi}((\cos(2\theta)+1) + 2\cos\theta - 3\sin^2\theta\cos\theta - \frac{1}{4}(1-\cos(4\theta))\Big)d\theta \\
&= \Big[\frac{1}{2}\sin(2\theta) + \theta + 2\sin\theta - \sin^3\theta - \frac{1}{4}\theta + \frac{1}{16}\sin(4\theta)\Big]_0^{2\pi} = \frac{3}{2}\pi.
\end{aligned} \tag{10}$$

Using (1), (5) and (10), we get

$$\oint_C -ydx + xdy = \frac{3}{2}\pi + \frac{3}{2}\pi = 3\pi.$$

$\square$

Finally we obtain by applying Green's theorem

$$\oint_C -ydx + xdy = \int\int_R \Big(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\Big)dxdy = \int\int_R 2dxdy = 2Area(R)$$

Hence the area of the region bounded by C is equal to

$$Area(R) = \frac{1}{2}\oint_C -ydx + xdy = \frac{3\pi}{2}.$$

$\square$

6. Evaluate the area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ that is within the planes $z = b$ and $z = c$, with $0 < b < c < a$.

Solution:

Let S be the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that is within the planes $z = b$ and $z = c$ (draw a figure). It is also defined by the equation $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$. Then the area of S is given by

$$Area(S) = \int\int_R \sqrt{1 + f_x^2 + f_y^2}dxdy, \tag{1}$$

where R is the projection of S on the $xy-$plane.
Now we have

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

We deduce then from (1) by using the polar coordinates and denoting by r_b and r_c the values of r for which we have respectively $z = b$ and $z = c$ i.e.

$r_b = \sqrt{a^2 - b^2}$ and $r_c = \sqrt{a^2 - c^2}$

$$\begin{aligned} Area(S) &= \int\int_R \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy \\ &= a \int_0^{2\pi} \int_{r_c}^{r_b} \frac{rdrd\theta}{\sqrt{a^2 - r^2}} \\ &= a \int_0^{2\pi} \big[-\sqrt{a^2 - r^2} \big]_{r_c}^{r_b} d\theta \\ &= 2\pi a(c - b). \end{aligned}$$

□

7. Evaluate the line integral $\oint_C yz^2dx + xz^2dy + x^2ydz$, where C is the trace of the cylinder $x^2 + z^2 = 1$ in the plane $z + y = 4$.

Solution:

Let $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + x^2y\mathbf{k}$ and let S be the portion of the plane $z + y = 4$ that is located inside the cylinder $x^2 + z^2 = 1$ (draw a figure). S is defined by the equation $z = f(x, y) = 4 - y$. The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. Therefore we have by Stokes' theorem

$$\oint_C yz^2dx + xz^2dy + x^2ydz = \oint_C \mathbf{F}.d\mathbf{r} = \int\int_S curl(\mathbf{F}).\mathbf{n}dS, \tag{1}$$

where

$$\begin{aligned} curl\mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & x^2y \end{vmatrix} \\ &= (x^2 - 2xz)\mathbf{i} - (2xy - 2yz)\mathbf{j} + (z^2 - z^2)\mathbf{k} \\ &= (x^2 - 2xz)\mathbf{i} - 2y(x - z)\mathbf{j} \end{aligned} \tag{2}$$

The unit normal vector to S is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \big(-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k} \big) = \frac{1}{\sqrt{2}} \big(\mathbf{j} + \mathbf{k} \big). \tag{3}$$

Now let R be the projection of S on the $xz-$plane. Then we get by using (1), (2) and (3)

$$\begin{aligned}
\oint_C yz^2dx + xz^2dy + x^2ydz &= \int\int_S -\sqrt{2}y(x-z)dS \\
&= \int\int_S -\sqrt{2}(4-z)(x-z)dS \\
&= \int\int_R -\sqrt{2}\sqrt{2}(4-z)(x-z)dxdz \quad \text{since } S \text{ is also defined by } y = 4-z \\
&= -2\int\int_R (4x - 4z - xz + z^2)dxdz. \qquad (4)
\end{aligned}$$

Using the polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we obtain

$$\begin{aligned}
&\int\int_R (4x - 4z - xz + z^2)dxdz \\
&= \int_0^{2\pi} \Big(\int_0^1 (4r\cos\theta - 4r\sin\theta - r^2\sin\theta\cos\theta + r^2\sin^2\theta)rdr \Big) d\theta \\
&= \int_0^{2\pi} \Big(\int_0^1 (4r^2\cos\theta - 4r^2\sin\theta - r^3\sin\theta\cos\theta + r^3\sin^2\theta)dr \Big) d\theta \\
&= \int_0^{2\pi} \Big(\Big[\frac{4}{3}r^3\cos\theta - \frac{4}{3}r^3\sin\theta - \frac{1}{4}r^4\sin\theta\cos\theta + \frac{1}{4}r^4\sin^2\theta\Big]_0^1 \Big) d\theta \\
&= \int_0^{2\pi} \Big(\frac{4}{3}\cos\theta - \frac{4}{3}\sin\theta - \frac{1}{4}\sin\theta\cos\theta + \frac{1}{4}\sin^2\theta \Big) d\theta \\
&= \int_0^{2\pi} \Big(\frac{4}{3}\cos\theta - \frac{4}{3}\sin\theta - \frac{1}{4}\sin\theta\cos\theta + \frac{1}{8}(1-\cos(2\theta)) \Big) d\theta \\
&= \Big[\frac{4}{3}\sin\theta + \frac{4}{3}\cos\theta - \frac{1}{8}\sin^2\theta + \frac{1}{8}\theta - \frac{1}{16}\sin(2\theta\Big]_0^{2\pi} = \frac{\pi}{4}. \qquad (5)
\end{aligned}$$

Hence we deduce from (1), (4) and (5) that

$$\oint_C yz^2dx + xz^2dy + x^2ydz = -\frac{\pi}{2}.$$

□

8. Evaluate $\int\int_S \mathbf{F}.\mathbf{n}\ dS$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the exterior surface of the region D that is above the $xy-$plane and below the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The volume of the region bounded by the ellipsoid is $\frac{4\pi}{3}abc$.

Solution:

The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. Therefore we have by the divergence theorem

$$\begin{aligned} \int\int_S \mathbf{F}.\mathbf{n}\ dS &= \int\int\int_D div(\mathbf{F})dV = \int\int\int_D (1+1+1)dV \\ &= \int\int\int_D 3dV = 3Vol(D) = 3\frac{1}{2}\frac{4\pi}{3}abc = 2\pi abc. \end{aligned}$$

□