Section 7.8 - Improper Integrals

Definition of Improper Integral of Type I:
(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ provided this limit exists.

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$ provided this limit exists.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limits exist and **divergent** if the limits do not exist.

Extension: If the improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{a}^{\infty} f(x)dx + \int_{-\infty}^{a} f(x)dx.$$

Definition of Improper Integral of Type II:

(a) If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$ if this limit exists.

(b) If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a, then $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$ if this limit exists.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Extension: If $f(x)$ has a discontinuity at c where $a < c < b$ and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example: Is $\int_1^2 \frac{1}{2x-1}dx$ an improper integral? State why or why not.

Solution:
The asymptote for the integrand is at $x = 1/2$. For interval $[1, 2]$, the integrand has no infinite discontinuity. In addition, the interval is not infinite. Hence, it is not an improper integral.

Example: Is $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}$ an improper integral? State why or why not.

Solution:
The integrand never becomes undefined within

the interval. Hence there is no infinite discontinuity. However, the interval is infinite. Thus, it is an improper integral.

Example: Determine if $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6} dx$ is convergent or divergent. Evaluate the integral if it is convergent.

Solution:
Let $u = x^3 \rightarrow du = 3x^2 dx$

$\int \frac{x^2}{9+x^6} dx = \int \frac{du}{3(u^2+9)} = [\frac{\tan^{-1}(x/3)}{9}]_{-\infty}^{\infty} = \frac{\pi}{9}$ since the tangent function goes to $\pm\infty$ at $x = \pm\pi/2$ (hence convergent).

Example: Determine if $\int_{-\infty}^{\infty} \frac{1}{x\sqrt{x}} dx$ is convergent or divergent. Evaluate the integral if it is convergent.

Solution:
$\int \frac{1}{x\sqrt{x}}dx = [\frac{-2}{\sqrt{x}}]_0^3$. We see that the function $\frac{-2}{\sqrt{x}}$ diverges at $x = 0$.