Section 7.4 - Integration of rational functions by partial fractions

Assuming that the integrands are of the following form: $S(x)+\frac{P(x)}{Q(x)}$

Case 1: Denominator $Q(x)$ is a product of distint linear factors.

If $Q(x)=(a_1x+b_1)(a_2x+b_2)\ldots(a_kx+b_k)$, then $\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\ldots+\frac{A_k}{ax_k+b_k}$ where $A_1,\ A_2,\ldots,A_k$ are constants. So the key is to find the constants $A_1,\ A_2,\ldots,A_k$.

Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.

If $Q(x)=(a_1x+b_1)^r$, then $\frac{R(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_1x+b_1)^2}+...\frac{A_r}{(a_1x+b_1)^r}$ where $A_1,\ A_2,\ldots,A_k$ are constants. So the key is to find the constants $A_1,\ A_2,\ldots,A_k$.

Illustration: $\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which are repeated.

Illustration: $\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+4}$

Case 4: $Q(x)$ contains a repeated irreducible quadratic factor.

Illustration: $\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x}+\frac{B}{x-1}$

$+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$

Notice that the highest order for the terms in the denominator (excluding the power) is **2**. This translates into the term in the numerator having the highest order of **1**. Can you see this?

Now let us go through some examples.

Example: Perform the partial fraction decomposition for $\frac{x^4}{(x^3+x)(x^2-x-3)}$

Solution:
The factors in the denominator are: x, (x^2+1), (x^2-x-3)
$\frac{x^4}{(x^3+x)(x^2-x-3)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2-x-1}$ because the highest order of x^2+x+1 and x^2-x-1 is 2.

Example: Evaluate $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2}dx$

Solution:
The integrand is now transformed: $\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$
Then, $x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-$

$3) = A(x^2+4x+4)+B(x^2-x-6)+C(x-3)$ lead to the following conditions:

coefficient of x^2: $1 = A + B$
coefficient of x: $0 = 4A - B + C$
coefficient of 1: $0 = 4A - 6B - 3C$

From second and third equations, we get: $4A-B+C = 4A-6B-3C \to 5B = -4C$ and $-9B + 16A = 0 \to 16A = 9B$

From first equation and using the latter relation, we get: $16A + 16B = 16 \to 25B = 16 \to B = 16/25$

Then, $A = 9B/16 = 9/25$ and $C = -5B/4 = -4/5$

So, $\int \frac{x^2}{(x-3)(x+2)^2} dx$

$= \int [\frac{9}{25(x-3)} + \frac{16}{25(x+2)} + \frac{(-4)}{5(x+2)^2}] dx$

$= \frac{9ln(x-3)}{25} + \frac{16ln(x+2)}{25} - \frac{8}{5(x+2)}$

Example: Evaluate $\int \frac{2x+1}{4x^2+12x-7} dx$

Solution:
$\frac{2x+1}{4x^2+12x-7} = \frac{2x+1}{4x^2+12x+9-16} = \frac{2x+1}{(2x+3)^2-4^2} = \frac{2x+1}{(2x-1)(2x+7)}$ $\frac{2x+1}{(2x-1)(2x+7)} = \frac{A}{2x-1} + \frac{B}{2x+7}$. This leads to:

$2 = 2A + 2B$
$1 = 7A - B$

Solving the above set of simultaneous equations and we end up with: $A = 1/4$ and $B = 3/4$

So, $\int \frac{2x+1}{4x^2+12x-7} dx = \frac{ln(2x-1)}{8} + \frac{3ln(2x+7)}{8}$