Section 7.2 - Trigonometric Integrals

There key points for this section can be summed up with 2 rules:

Suppose m, n, k are integers.

Rule 1: Evaluating integrals $\int \sin^m x \cos^n x dx$
(a) if $n = 2k+1$(i.e. odd), then use identity: $\cos^2 x + \sin^2 x = 1$ to re-order $\cos^n x$

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x (1 - \sin^2 x)^k \cos x dx$$

Can then use integration by substitution by letting $u = \sin x$

(b) If $m = 2k+1$ (i.e. odd), then use identity: $\cos^2 x + \sin^2 x = 1$ to re-order $\sin^m x$

$$\int \sin^m x \cos^{2k+1} x dx = \int (1 - \cos^2 x)^k \cos^n x \sin x dx$$

Can then use integration by substitution by letting $u = \cos x$.

(c) If both m and n are even then we can use the half angle identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and the identity: $\sin 2x = \frac{\sin x \cos x}{2}$.

Rule 2: Evaluating integrals $\int \tan^m x \sec^n x dx$

(a) If $n = 2k$ (i.e. even), then use the identity: $\sec^2 x = 1 + \tan^2 x$ to re-order the $\sec^n x$ term:

$$\int \tan^m x \sec^{2k} x dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx$$

Can then use integration by substitution by letting $u = \tan x$.

(b) If $m = 2k + 1$ (i.e. odd), then use the identity: $\sec^2 x = 1 + \tan^2 x$ to re-order the $\tan^m x$ term:

$$\int \tan^m x \sec^n x dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx$$

Can then use integration by substitution by letting $u = \sec x$.

It may be necessary to use the following identities when evaluating integrals of the form $\int \sin(mx)\cos(nx)$, $\int \sin(mx)\sin(nx)$ and $\int \cos(mx)\cos(nx)$.

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \sin(A+B)]$

Example: Evaluate $\int_0^{\pi/3} \tan^5 x \sec^6 x dx$

Solution:
Here, the power of $\tan x$ is odd, so we use

$k = 2,\ n = 6.$

The integral becomes:
$\int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^5 x \sec x \tan x dx = \int_0^{\pi/3} (\sec^4 x - 2\sec^2 x + 1)\sec^5 x(\sec x \tan x) dx = \int_0^{\pi/3} (\sec^9 x - 2\sec^7 x + \sec^5 x)\sec x \tan x dx$

So if $u = \sec x$, then $du = \sec x \tan x dx$ and

$= [\frac{\sec^{10} x}{10} - \frac{\sec^8 x}{4} + \frac{\sec^6 x}{6}]_0^{\pi/3}$