Section 5.5 - The Substitution Rule

Rule 1: If $u = g(x)$ is a differentiable function whose range is an interval I and $f(x)$ is continuous on I then $\int f(g(x))g'(x)dx = \int f(u)du$

Example: Evaluate the integral $\int \frac{x}{(x^2+1)^2}dx$

Solution:
let $u = x^2 + 1$, then $du = 2xdx \to \frac{du}{2} = dx$.
$$\int \frac{x}{(x^2+1)^2}dx = \int \frac{du}{2u^2} = -\frac{1}{2(x^2+1)}$$

Rule 2: If $g'(x)$ is continuous on $[a,b]$ and $f(x)$ is continuous on the range of $u = g(x)$ then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Example: If $f(x)$ is continuous and $\int_0^9 f(x)dx = 4$, then find $\int_0^3 xf(x^2)dx$

Solution:
let $u = x^2$, then $\frac{du}{2} = xdx$.

$$\int_0^3 xf(x^2)dx = \frac{1}{2}\int_{0^2}^{3^2} f(u)du = 2.$$

Rule 3: Suppose $f(x)$ is continuous on $[-a, a]$.

(a) If $f(x)$ is even $[f(-x) = f(x)]$ then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$. (b) If $f(x)$ is odd $[f(-x) = -f(x)]$ then $\int_{-a}^{a} f(x)dx = 0$.

Example: Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 sin(x)}{1+x^6}dx$

Solution:
Because $\frac{x^2 sin(x)}{1+x^6}$ is an odd function (due to $sin(x)$), the integral is zero.