Section 5.3 - Fundamental Theorem of Calculus

Suppose $f(x)$ is continuous on $[a, b]$.

- If $g(x) = \int_a^x f(t)\ dt$, then $g'(x) = f(x)$.

- $\int_a^b f(x)\ dx = F(b) - F(a)$ where $F(x)$ is the anti-derivative of $f(x)$, i.e. $F'(x) = f(x)$

Example: Use the Fundamental Theorem of Calculus to find the derivative of $g(x) = \int_1^x ln(t)dt$.

Solution:
As $ln(t)$ is continuous from 1 to x where $x \geq 1$. The FTC then says that $g'(x) = ln(x)$.

Example: Use the Fundamental Theorem of Calculus to evaluate the integral $\int_{-5}^{5} \frac{2}{x^3} dx$ or explain why it does not exist.

Solution:
The integral does not exist because the integrand $\frac{2}{x^3}$ is not continuous in the domain $[-5, 5]$. There is a discontinuity at $x = 0$ since the integrand is undefined there.

Example: Find the derivative of the function

$$g(x) = \int_{tan(x)}^{x^2} \frac{1}{\sqrt{2 + t^4}}$$

Solution:
To do this problem, we need to apply the chain rule in addition to the Fundamental Theorem of Calculus. Suppose $f(x) = \frac{1}{\sqrt{2+x^4}}$.

If $F'(x) = f(x)$ and let $\alpha(x) = tan(x)$ and $\beta(x) = x^2$, then $g(x) = F(\beta(x)) - F(\alpha(x))$.

$$g'(x) = F'(\beta(x)) \times \beta'(x) - F'(\alpha(x)) \times \alpha'(x)$$

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$$g'(x) = \frac{2x}{\sqrt{2+x^8}} - \frac{sec^2(x)}{\sqrt{2+tan^4(x)}}$$