Section 11.6 - Absolute Convergence and the Ratio and Root Tests

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Ratio Test:

(i) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent.

(ii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the Ratio Test is inconclusive.

Root Test:

(i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent.

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

Example: Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}$ is absolutely convergent, conditionally convergent or divergent.

Let $a_n = (-1)^n b_n$ and $b_n = \frac{1}{n^{1/4}}$. Since $\lim_{n\to\infty} b_n = 0$ and b_n is decreasing, we have convergence using the alternating series test. However, $\sum_{n=1}^{\infty} |a_n|$ diverges by the p-test. Hence, it is conditionally convergent.

Example: Determine whether the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$$
is absolutely convergent, conditionally convergent or divergent.

Let $a_n = (-1)^n b_n$ and $b_n = \frac{e^{1/n}}{n^3}$. $\lim_{n\to\infty} b_n = 0$ and $b_n' < 0 \Rightarrow a_n$ is decreasing. Hence, covergence by the alternating series test.

$\lim_{n\to\infty} |(-1)^n a_n| = 0$ because $e^{1/n} < n^3$ for large n. Hence, it is absolutely convergent.

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ is absolutely convergent, conditionally convergent or divergent.

Let $a_n = \frac{n^n}{3^{1+3n}}$. Using ratio test, $|\frac{a_{n+1}}{a_n}| = \frac{1}{3^3} \cdot (\frac{(n+1)}{n})^n \cdot (n+1)$. We see that as we take the limit as $n \to \infty$, $|\frac{a_{n+1}}{a_n}| \to \infty$ and hence, the series is divergent.

Example: Determine whether the series $\sum_{n=1}^{\infty} (\frac{n^2+1}{2n^2+1})^n$ is absolutely convergent, conditionally convergent or divergent.

Let $a_n = (\frac{n^2+1}{2n^2+1})^n$. Using the root test, we have $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$. Hence the series is absolutely convergent.