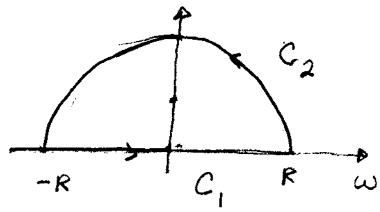
HW # 3, Problem (5)

Use contour integration to find $\mathcal{F}^{-1}\left\{\frac{\omega}{\omega^2+1}\right\}$.

<u>soln.</u>

By defn. we have $f(t) = \mathcal{F}^{-1}\left\{\frac{\omega}{\omega^2+1}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega e^{i\omega t}}{1+\omega^2}\, d\omega.$

Consider the integral $I = \int_C \frac{z e^{izt}}{1+z^2}\, dz$ on a closed contour C.



<u>1) $t>0$.</u> Let $C = C_1 \cup C_2$ as shown.

On C_2, $z = Re^{i\theta}$, $0 \le \theta \le \pi$, and

$$\left|\frac{z}{1+z^2}\right| = \left|\frac{Re^{i\theta}}{1+R^2 e^{2i\theta}}\right| < \frac{R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

$\therefore$ By Jordon's lemma, $\int_{C_2} \to 0$ as $R \to \infty$.

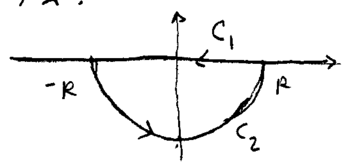
On C_1, $z = x$ and $\int_{C_1} \to 2\pi f(t)$.

$\therefore \int_C \to 2\pi f(t)$

The integrand has a simple pole at $z=i$ inside C.

$$\text{Res}(z=i) = \lim_{z\to i}(z-i)\frac{z e^{izt}}{z^2+1} = \lim_{z\to i}\frac{z e^{izt}}{z+i} = e^{-t}/2.$$

Therefore, $2\pi i\, e^{-t}/2 = 2\pi f(t) \Rightarrow f(t) = i e^{-t}/2.$



2) $t<0$ Let $C = C_1 \cup C_2$ as shown.

As above, we can show using Jordan's lemma that $\int_{C_2} \to 0$ as $R \to \infty$.

Also, $\int_{C_1} \to -2\pi f(t)$. Again, the integrand has a simple pole at $z=-i$ inside C and

$$\text{Res}(z=-i) = \lim_{z\to -i}\frac{z e^{izt}}{z-i} = e^{t}/2.$$

Thus $\int_C = 2\pi i e^t/2 = -2\pi f(t) \Rightarrow f(t) = -ie^t/2.$

$$\therefore\quad f(t) = \frac{i}{2}\begin{cases} e^{-t} & t>0 \\ -e^{t} & t<0 \end{cases} = \frac{i}{2}\left\{H(t)e^{-t} - H(-t)e^{t}\right\}.$$

or we can write:

$$f(t) = \frac{i}{2} e^{-|t|}\begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases} = \frac{i}{2} e^{-|t|}\,\text{sgn}(t).$$