Math-260 (022)

A Solution Of Exam II

1. $A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$. Then $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}b = \frac{1}{10}\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow X = \begin{bmatrix} 0.3 \\ -1.1 \end{bmatrix}$. $x_1 = 0.3$ and $x_2 = -1.1$.

---

2. Given $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. We find A^{-1}

$$[A : I]_{3\times 3} = \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -7 & 1 & -2 & 0 \end{array}\right]$$

$$\underset{\sim}{7R_2+R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 7 & 0 & 1 & -2 & 7 \\ 0 & 0 & -7 & 1 & -2 & 0 \end{array}\right] \underset{\frac{1}{-7}R_3}{\overset{\frac{1}{7}R_2}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{7} & \frac{-2}{7} & 1 \\ 0 & 0 & 1 & \frac{-1}{7} & \frac{2}{7} & 0 \end{array}\right]$$

$$\underset{\sim}{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 1 & 0 & \frac{1}{7} & -\frac{2}{7} & 1 \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{3}{7} & 1/7 & 0 \\ \frac{1}{7} & -2/7 & 1 \\ -1/7 & 2/7 & 0 \end{bmatrix}$$

---

3. Given $v_1 = (3,1,0)$, $v_2 = (0,1,0)$, and $v_3 = (-1,0,0)$.

These vectors are linearly dependent as:

$$v_1 = v_2 - 3v_3$$

and thus they do not form a basis for $\mathbb{R}^3$.

Another solution
$|v_1\ v_2\ v_3| = 0$.

---

4. The set $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 . x_2 = 0\}$ is not a subspace as it does not satisfy the closure-property in addition,

for example: $w_1 = (1,0,0,0)$ and $w_2 = (0,1,0,0)$ are in W

but $w_1 + w_2 = (1,1,0,0) \notin W$ because $1.1 = 1 \neq 0$.

---

5. We find a basis for the solution space of the system:

$$(1)\quad \begin{cases} x_1 - 3x_2 - x_3 = 0 \\ x_1 + 3x_2 - 3x_3 = 0 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right) \overset{-R_1+R_2}{\sim} \left(\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 6 & -2 & 0 \end{array}\right)$$

The leading variables are x_1 and x_2 while x_3 is a free variable. Put $\boxed{x_3 = t}$, $t \in \mathbb{R}$.

$$\Rightarrow 6x_2 = 2x_3 \Rightarrow \boxed{x_2 = \tfrac{1}{3}t}$$

Similarly, $x_1 - 3x_2 - x_3 = 0 \Rightarrow x_1 = 3(\tfrac{1}{3}t) + t \Rightarrow \boxed{x_1 = 2t}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 2 \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

$\therefore$ Any solution of (1) is a scalar multiplication of $\begin{pmatrix} 2 \\ \frac{1}{3} \\ 1 \end{pmatrix}$.

That is Solution Space $= \text{Span}\left\{ \begin{pmatrix} 2 \\ \frac{1}{3} \\ 1 \end{pmatrix} \right\}$.

$\Rightarrow \left\{ \begin{pmatrix} 2 \\ \frac{1}{3} \\ 1 \end{pmatrix} \right\}$ is a basis for the solution space of our system.

---

6. We show that $y = x^3$ is a solution of the differential equation: $x^3 y^{(3)} - 3x^2 y'' + 6x y' - 6y = 0$. (2)

First: $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow y'' = 6x \Rightarrow y^{(3)} = 6$.

Substitution in (2) results: $x^3(6) - 3x^2(6x) + 6x(3x^2) - 6x$

$$= (6 - 18 + 18 - 6)x^3 = 0.$$

$\Rightarrow y = x^3$ is a solution of (2).

---

7. We verify: $f(x) = e^x$ and $g(x) = \sin x$ are linearly independent on $[0, 10]$ by two methods.

- Method 1: We show that $W(f,g) \neq 0$ at $\underline{\text{some}}$ point $x \in [0, 1]$. $W(f,g) = \begin{vmatrix} e^x & \sin x \\ e^x & \cos x \end{vmatrix} = e^x(\cos x - \sin x)$

Then $W(f,g) = 1 \neq 0$ when $x = 0 \Rightarrow f$ & g are linearly independ

Method 2: Since $\dfrac{f(x)}{g(x)} = \dfrac{e^x}{\sin x} \neq$ constant $\Rightarrow f \,\&\, g$ are linearly Independent.

---

8. We solve: $2y'' + 4y' = 0$ with $y(1) = -1$, $y'(1) = 0$.

The characteristic equation is: $2r^2 + 4r = 0 \Rightarrow 2r(r+2) = 0$

$\Rightarrow r = 0$ or $r = -2 \Rightarrow$ The general solution: $\boxed{y(x) = c_1 + c_2 e^{-2x}}$.

Now: $\because$ $y(1) = -1 \Rightarrow -1 = c_1 + c_2 e^{2}$

and $\because$ $y'(1) = 0$, $y'(x) = -2c_2 e^{-2x} \Rightarrow 0 = -2c_2 e^{-2} \Rightarrow \boxed{c_2 = 0}$

$\Rightarrow \boxed{c_1 = -1} \Rightarrow \boxed{y(x) = -1}$

---

9. We find a particular solution of:

$$y'' - y = e^{2x} - x \qquad (1)$$

Consider y_p in the form: $y_p = Ae^{2x} + Bx$

$$\Rightarrow y_p' = 2Ae^{2x} + B$$

$$\Rightarrow y_p'' = 4Ae^{2x}$$

Substituting y_p & y_p'' in (1) gives:

$$4Ae^{2x} - Ae^{2x} - Bx = e^{2x} - x$$

$$\Rightarrow 3Ae^{2x} - Bx = e^{2x} - x$$

$$\Rightarrow 3A = 1 \quad \text{and} \quad -B = -1$$

$$\Rightarrow \boxed{A = \tfrac{1}{3}} \quad \text{and} \quad \boxed{B = 1}$$

Then $\boxed{y_p(x) = \frac{1}{3} e^{2x} + 1}$.

10: Since $(D-2)^3 y = 0$

$\Rightarrow$ Ch. equation is: $(r-2)^3 = 0$

and the general solution is:

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$$

$$= e^{2x}(c_1 + c_2 x + c_3 x^2)$$

---

11. The ch. equation of:

$y''' - 2y'' + 2y' = 0$ is

$r^3 - 2r^2 + 2r = 0 \Rightarrow r(r^2 - 2r + 2) = 0$

$\Rightarrow r = 0$ or $r = \dfrac{2 \pm \sqrt{4-8}}{2}$

$\Rightarrow r = 0$ or $r = 1 \pm i$

$\Rightarrow$ the general solution is:

$$y(x) = c_1 + e^x(c_2 \cos x + c_3 \sin x).$$