1. (8 points) Find the general solution of the differential equation

$$(1-x^2)y'' + 2xy' - 2y = 0 \qquad (x > 1)$$

given that $y_1 = x$ is a solution of the differential equation.

Rewrite the DE as $\left\{ y'' + \frac{2x}{1-x^2} y' - \frac{2}{1-x^2} y = 0 \right\}$ 1 .

With $p(x) = \frac{2x}{1-x^2}$, a second solution is

$\left\{ y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \right\}$, $\quad -\int p(x)\,dx = -\int \frac{2x}{1-x^2}\,dx$
$= \ln(x^2-1)$

So, $y_2 = x \int \frac{e^{\ln(x^2-1)}}{x^2}\,dx = x \int \frac{x^2-1}{x^2}\,dx = x\int\left(1 - \frac{1}{x^2}\right)dx$

$= x\left(x + \frac{1}{x}\right) = x^2 + 1.$

y_1 and y_2 are linearly independent, they form a fundamental set of solutions of the DE.

The general solution is : $y = C_1 x + C_2(x^2+1)$.

2. (8 points) Find the general solution of the differential equation

$$y^{(4)} - 2y''' + 5y'' = 0.$$

Solution: Let $y = e^{mx}$

The auxiliary equation is:

$m^4 - 2m^3 + 5m^2 = 0 \Rightarrow m^2\underbrace{(m^2 - 2m + 5)}_{} = 0$

$m_1 = m_2 = 0$ $\qquad m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$m_3 = 1 + 2i$

$m_4 = 1 - 2i$

The general solution is:

$$y = C_1 + C_2 x + e^{x}\left(C_3 \cos(2x) + C_4 \sin(2x)\right).$$

3. (9 points) Consider the differential equation

$$x^4y'' + x^3y' - 4x^2y = 1 \qquad (x > 0).$$

(a) Given that $y_1 = x^2$ and $y_2 = x^{-2}$ are solutions of the associated homogeneous equation. Show that they form a fundamental set of solutions of the associated homogeneous equation.

(b) Find a particular solution of the given non-homogeneous equation.

a) $W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1} \neq 0$

$\therefore$ y_1 and y_2 are l. i.

$\therefore$ $\{y_1, y_2\}$ is a fundamental set

b) $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = x^{-4} = f(x)$

Let

$y_p = u_1 y_1 + u_2 y_2 = x^2 u_1 + x^{-2} u_2$

$W = -4x^{-1} = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix}$

$W_1 = \begin{vmatrix} 0 & x^{-2} \\ x^{-4} & -2x^{-3} \end{vmatrix} = -x^{-6}$

$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & x^{-4} \end{vmatrix} = x^{-2}$

$u_1' = W_1/W = \frac{(-x^{-6})}{-4x^{-1}} = \frac{1}{4}x^{-5}$, $u_2' = \frac{x^{-2}}{-4x^{-1}} = \frac{-x^{-1}}{4}$

$u_1 = \frac{-x^{-4}}{16}$, $u_2 = -\frac{1}{4}\ln x$

$\therefore$ $y_p = x^2\left(\frac{-x^{-4}}{16}\right) + x^{-2}\left(-\frac{1}{4}\ln x\right) = -\frac{1}{16}x^{-2} - \frac{1}{4}x^{-2}\ln x$

$= -\frac{x^{-2}}{16}(1 + 4\ln x)$

4. **(7 points)** Find the general solution of the differential equation

$$x^2y'' + 11xy' + 41y = 0 \qquad (x > 0).$$

This is a Cauchy-Euler equation.

We look for solutions of the form $y = x^m$.

This gives $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$.

We get $\{ m(m-1)x^m + 11mx^m + 41x^m = 0 \}$.

i.e. $m(m-1) + 11m + 41 = 0$

or $\{ m^2 + 10m + 41 = 0 \}$

$(m+5)^2 + 16 = 0$

with roots: $\{ m_1 = -5+4i,\ m_2 = -5-4i \}$

The general solution is therefore

$$y = x^{-5}\left(C_1 \cos(4\ln x) + C_2 \sin(4\ln x)\right)$$

5. (8 points) Find the recurrence relation for the coefficients of power series solutions of $y'' + 2xy' + 2y = 0$ about the ordinary point $x = 0$.

$x=0$ is an ordinary point of the DE.

$$y = \sum_{k\geq 0} a_k x^k$$

$$y' = \sum_{k\geq 1} k a_k x^{k-1}$$

$$y'' = \sum_{k\geq 2} k(k-1) a_k x^{k-2}$$

replacing into the DE, we get

$$\sum_{k\geq 2} k(k-1)a_k x^{k-2} + 2\cancel{x} \sum_{k\geq 1} k a_k x^{k-1} + 2 \sum_{k\geq 0} a_k x^k \equiv 0$$

(let $k-2=l$)

$$\sum_{l\geq 0} (l+2)(l+1)a_{l+2} x^l + \sum_{l\geq 1} 2 l a_l x^l + \sum_{l\geq 0} 2 a_l x^l \equiv 0$$

$$2a_2 + 2a_0 + \sum_{l\geq 1} \left[(l+2)(l+1)a_{l+2} + 2 l a_l + 2a_l\right] x^l \equiv 0$$

hence, $2a_2 + 2a_0 = 0$ ie $\underline{a_2 = -a_0}$

$$(l+2)(l+1)\, a_{l+2} + 2(l+1)a_l = 0, \quad l \geq 1$$

ie

$$\boxed{a_{l+2} = -\frac{2a_l}{l+2}, \quad l\geq 1}$$

recurrence relation.

6. (10 points) By substituting $y = \sum_{n=0}^{\infty} c_n x^n$ in a differential equation, we obtain

$$2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty} [(k+1)(k-1)c_k - (k+2)(k+1)c_{k+2}]x^k = 0, \quad \text{for all } x.$$

Find the general solution of that differential equation.

The conditions $2c_2 - c_0 = 0$ & $6c_3 = 0$

$c_2 = \frac{1}{2} c_0 \qquad c_3 = 0$

The recurrence relation $(k+1)(k-1)c_k - (k+2)(k+1)c_{k+2} = 0$. So,

$$c_{k+2} = \frac{k-1}{k+2} c_k, \quad k = 2, 3, 4, \ldots$$

$k=2$; $c_4 = \frac{1}{4} c_2 = \frac{1}{2 \cdot 4} c_0$

$k=3$; $c_5 = \frac{2}{5} c_3 = 0$

$k=4$; $c_6 = \frac{3}{6} c_4 = \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} c_0$

$k=5$; $c_7 = \frac{4}{7} c_5 = 0$

$k=6$; $c_8 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} c_0$

$\Rightarrow$ The solution is $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^{\cancel{6}7} + c_8 x^8 + \cdots$

$= c_0 + c_1 x + \frac{1}{2} c_0 x^2 + \frac{1}{2 \cdot 4} c_0 x^4 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} c_0 x^6 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} c_0 x^8 + \cdots$

$= c_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} x^6 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^8 + \cdots\right) + c_1 (x)$

$y_1 = x$; $y_2 = 1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} x^6 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^8 + \cdots$

$$y_2 = 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n (n!)} x^{2n}$$

The general solution is $y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 \left(1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n (n!)} x^{2n}\right)$