1. (a) (2 points) Verify that $e^y = y - x^2 + C$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{2x}{1 - e^y}.$$

(b) (2 points) Use the implicit solution given in part (a) to solve the initial-value problem (IVP)

$$\frac{dy}{dx} = \frac{2x}{1 - e^y}, \quad y(1) = 1.$$

(c) (2 points) Tell whether the IVP given in part (b) has a unique solution. Justify your answer.

(a) $e^y = y - x^2 + C \Rightarrow y' e^y = y' - 2x$

$\Rightarrow y'(e^y - 1) = -2x \Rightarrow y' = \dfrac{2x}{1 - e^y}$

So, $e^y = y - x^2 + C$ is a solution of the given DE.

(b) $e^y = y - x^2 + C$ is a solution of the IVP if the initial condition is satisfied, that is,

$e^1 = 1 - 1^2 + C \Rightarrow \boxed{C = e}$, so

$e^y = y - x^2 + e$ is a solution of the IVP.

(c) $e^y = y - x^2 + e$ is the unique solution of the given IVP on some interval of definition I centered at 1, because $f(x,y) = \dfrac{2x}{1-e^y}$ and $\dfrac{\partial f}{\partial y} = \dfrac{2e^y x}{(1-e^y)^2}$ are continuous on some region contained the point $(1,1)$ in its interior.

2. (7 points) Solve: $\sin x \frac{dy}{dx} - y\cos x = \sin^2 x$, where $0 < x < \frac{\pi}{2}$.

$\div \sin x$

$$\Rightarrow \frac{dy}{dx} - y\cot x = \sin x$$

(Linear)

Now, $P(x) = -\cot x$ and

$$I = e^{\int -\cot x\,dx} = e^{-\int \frac{\cos x}{\sin x}dx}$$

$$= e^{-\ln|\sin x|} = e^{\ln \csc x} = \csc x$$

So, the DE becomes

$$\csc x \frac{dy}{dx} - y \csc x \cot x = 1$$

$$\Rightarrow \frac{d}{dx}(y \csc x) = 1$$

$$\Rightarrow y \csc x = x + c$$

$$\Rightarrow y = x \sin x + c \sin x$$

3. (7 points) Solve: $e^{x^3+y^2}\,dx + \dfrac{y}{x^2}\,dy = 0$.

$$\underline{e^{x^3} \cdot e^{y^2}}\, dx = \underline{-\frac{y}{x^2}}\, dy$$

Separates: $\underline{-x^2 e^{x^3}}\,dx = \underline{y e^{-y^2}}\,dy$

Integrate to get: $\underline{\frac{-1}{3}e^{x^3}} = \underline{C} - \underline{\frac{1}{2}e^{-y^2}}$

$\left(\text{or } \frac{e^{-y^2}}{2} - \frac{e^{x^3}}{3} = C \text{ or } 2e^{x^3} - 3e^{-y^2} = C\right)$

4. (7 points) Solve: $(ye^x + \sin x)dx + (2y + e^x + \cos y)dy = 0$.

$M = ye^x + \sin x$, $N = 2y + e^x + \cos y$

$M_y = e^x = N_x$, so the DE is exact.

Thus, there exists a function f such that $df = M\,dx + N\,dy$.

So, $\begin{cases} f_x = ye^x + \sin x & (a) \\ f_y = 2y + e^x + \cos y & (b) \end{cases}$

Integrating (a), $f = ye^x - \cos x + g(y)$

differentiate w.r.t. y and compare the results with (b),

$$e^x + g'(y) = 2y + e^x + \cos y$$

$$\Rightarrow g'(y) = 2y + \cos y \Rightarrow g(y) = y^2 + \sin y + c_1$$

$$\therefore f = ye^x - \cos x + y^2 + \sin y + C_1 \qquad (*)$$

On the other hand, $df = 0$, so $f = C_2$ $\quad (**)$

Combine $(*)$ and $(**)$, $ye^x - \cos x + y^2 + \sin y = C$ is a solution of the given DE.

5. (6 points)

(a) Use an appropriate substitution to reduce the following differential equation

$$\frac{dy}{dx} = \frac{2y - x}{x + 3y}$$

to a separable equation.

(b) Is it possible to write the separable equation obtained in (a) as a linear differential equation? Justify your answer.

(a) $\frac{dy}{dx} = \frac{2y - x}{x + 3y} = \frac{2\frac{y}{x} - 1}{3\frac{y}{x} + 1}$

Let $u = \frac{y}{x}$, that is $y = ux$

So $dy = u\,dx + x\,du$

The equation becomes

$$u + x\frac{du}{dx} = \frac{2u - 1}{3u + 1}$$

or $x\frac{du}{dx} = \frac{2u-1}{3u+1} - u = \frac{-3u^2 + u - 1}{3u + 1}$

Thus, $\frac{3u+1}{-3u^2+u-1}\,du = \frac{dx}{x}$ — (*)

which is a separable differential equation.

(b) If we use x as a function of u, we can write (*) as $\frac{dx}{du} - \frac{3u+1}{-3u^2+u-1}\,x = 0$

which is a linear equation.

6. (7 points) According to Newton's Law of cooling/warming, the rate of change of temperature $T(t)$ of an object at any time t is proportional to the difference between T and the surrounding temperature T_m. Let k be the constant of proportionality.

(a) Write the differential equation that models this phenomenon.

(b) Solve the differential equation found in (a) and write its general solution as $T(t) = T_m + ce^{kt}$.

(c) An object of temperature 10°C is left in a room of temperature 30°C. After 2 minutes the object temperature is 15°C. How long will it take for the object to reach 25°C?

a) $\frac{dT}{dt} = k(T - T_m)$

b) $\frac{dT}{T - T_m} = k\,dt \Rightarrow \ln|T - T_m| = kt + c_1$

$\Rightarrow |T - T_m| = c_2 e^{kt} \Rightarrow T - T_m = ce^{kt}$

or $T = T_m + ce^{kt}$

c) $T(0) = 10$, $T_m = 30$, $T(2) = 15$

$\Rightarrow 10 = 30 + c \Rightarrow c = -20$

$T(t) = 30 - 20e^{kt}$

$15 = 30 - 20e^{2k} \Rightarrow e^{2k} = \frac{-15}{-20} = 3/4$

$\Rightarrow k = \frac{1}{2}\ln 3/4$

$25 = 30 - 20e^{\frac{1}{2}\ln 3/4\, t}$

$\Rightarrow \frac{-5}{-20} = e^{\frac{1}{2}\ln 3/4\, t} \Rightarrow \frac{1}{4} = e^{\frac{1}{2}\ln 3/4\, t}$

$\Rightarrow \ln 1/4 = \frac{1}{2}\ln\frac{3}{4}\,t \Rightarrow t = \frac{2\ln 1/4}{\ln 3/4}$